

THE DEFORMATION OF SYMPLECTIC CRITICAL SURFACES IN A KÄHLER SURFACE-I

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ABSTRACT. In this paper we derive the Euler-Lagrange equation of the functional $L_\beta = \int_\Sigma \frac{1}{\cos^\beta \alpha} d\mu$, $\beta \neq -1$ in the class of symplectic surfaces. It is $\cos^3 \alpha \mathbf{H} = \beta(J(J\nabla \cos \alpha)^\top)^\perp$, which is an elliptic equation when $\beta \geq 0$. We call such a surface a β -symplectic critical surface. We first study the properties for each fixed β -symplectic critical surface and then prove that the set of β where there is a stable β -symplectic critical surface is open. We believe it should be also closed. As a precise example, we study rotationally symmetric β -symplectic critical surfaces in \mathbb{C}^2 carefully.

1. INTRODUCTION

Suppose that M is a Kähler surface. Let ω be the Kähler form on M and let J be a complex structure compatible with ω . The Riemannian metric \langle, \rangle on M is defined by

$$\langle U, V \rangle = \omega(U, JV).$$

For a compact oriented real surface Σ which is smoothly immersed in M , one defines, following [3], the Kähler angle α of Σ in M by

$$(1.1) \quad \omega|_\Sigma = \cos \alpha d\mu_\Sigma$$

where $d\mu_\Sigma$ is the area element of Σ of the induced metric from \langle, \rangle . We say that Σ is a holomorphic curve if $\cos \alpha \equiv 1$, Σ is a Lagrangian surface if $\cos \alpha \equiv 0$ and Σ is a symplectic surface if $\cos \alpha > 0$.

In [4] we consider the functional

$$L = \int_\Sigma \frac{1}{\cos \alpha} d\mu.$$

The Euler-Lagrange equation of this functional is

$$\cos^3 \alpha \mathbf{H} = (J(J\nabla \cos \alpha)^\top)^\perp.$$

We call such a surface a symplectic critical surface. We study the properties of the symplectic critical surfaces. In this paper, we consider a sequence of functionals

$$L_\beta = \int_\Sigma \frac{1}{\cos^\beta \alpha} d\mu.$$

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The critical point of the functionals L_β in the class of symplectic surfaces in a Kähler surface is called a β -symplectic critical surface. We first calculate the Euler-Lagrange equation of L_β .

Theorem 1.1. *Let M be a Kähler surface. The Euler-Lagrange equation of the functional L_β ($\beta \neq -1$) is*

$$(1.2) \quad \cos^3 \alpha \mathbf{H} - \beta (J(J\nabla \cos \alpha)^\top)^\perp = 0,$$

where H is the mean curvature vector of Σ in M , and $()^\top$ means tangential components of $()$, $()^\perp$ means the normal components of $()$.

Similar to the case that $\beta = 1$ in [4], we can check that it is an elliptic equation module tangential diffeomorphisms if $\beta \geq 0$.

The equation (1.2) seems quit interesting. It is the minimal surface equation as $\beta = 0$. In the case that $\beta \neq 0$, a minimal surface with constant Kähler angle (an infinitesimally holomorphic immersion) satisfies the equation, especially two kinds of important surfaces, i.e. holomorphic curves and special Lagrangian surfaces satisfy the equation. However there are many β -symplectic critical surfaces which are not minimal surfaces.

Existence of holomorphic curves in a Kähler surface is a fundamental problem in differential geometry. It is known that a closed symplectic minimal surface in a compact Kähler-Einstein surface with nonnegative scalar curvature is holomorphic ([9]). On the contrary, C. Arezzo ([1]) constructed examples which shows that a strictly stable minimal surface in a Kähler-Einstein surface with negative scalar curvature may not be holomorphic.

Our goal in this paper is to start a program to deform β -symplectic critical surface from a minimal surface ($\beta = 0$) to a holomorphic curve ($\beta = \infty$) by using continuity method. Actually, one can check that if we can deform the β -symplectic critical surface Σ_β for $\beta \in [0, \infty)$ with uniformly bounded $L_\beta(\Sigma_\beta)$ to a limit Σ_∞ smoothly, then Σ_∞ is a holomorphic curve in M , regardless of the sign of the scalar curvature of M . We do not need M to be a Kähler-Einstein surface.

We first examine the properties of β -symplectic critical surfaces for each fixed β . We will see that β -symplectic critical surfaces share many properties with minimal surfaces (c.f. [3], [4], [5], [9],[10]). We derive an equation for the Kähler angle of a β -symplectic critical surface in a Kähler-Einstein surface.

Theorem 1.2. *If M is a Kähler-Einstein surface and Σ is a β -symplectic critical surface, then we have*

$$\Delta \cos \alpha = \frac{2\beta \sin^2 \alpha}{\cos \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)} |\nabla \alpha|^2 - 2 \cos \alpha |\nabla \alpha|^2 - \frac{K}{4} \frac{\cos^3 \alpha \sin^2 \alpha}{\cos^2 \alpha + \beta \sin^2 \alpha},$$

where K is the scalar curvature of M .

The theorem yields that a β -symplectic critical surface in a Kähler-Einstein surface with nonnegative scalar curvature is holomorphic.

Then we calculate the second variation formula of the functionals L_β . As a corollary (c.f. [4], [5]), we show that

Theorem 1.3. *Let M be a Kähler surface with positive scalar curvature R . If Σ is a stable β -symplectic critical surface in M with $\beta \geq 0$ and $\chi(\nu) \geq g$, where $\chi(\nu)$ is the Euler characteristic of the normal bundle ν of Σ in M and g is the genus of Σ , then Σ is a holomorphic curve.*

In order to proceed the continuity method, we define the set

$$S := \{\beta \in [0, \infty) \mid \exists \text{ a strictly stable } \beta\text{-symplectic critical surface}\}.$$

As a first attempt to the continuity method and as an application of the second variation formula, we prove that

Theorem 1.4. *The set S is open in $[0, \infty)$.*

The existence part will depend on a theorem proved by B. White ([8]) combining with Implicit Function Theorem. White's theorem tells us that the Jacobi operator is a Fredholm map with Fredholm index 0. The stability assumption implies that the Jacobi operator is injective, thus isomorphism. The stability of the solution follows from the continuity of the Jacobi operator with respect to β . Of course, the closeness (or, the compactness) part is more delicate. We will attach this problem in a subsequent paper.

As a verification of our idea, we study the rotationally symmetric graphic β -symplectic critical surface in \mathbb{C}^2 which is given by

$$F(r, \theta) = (r \cos \theta, r \sin \theta, f(r), g(r)).$$

We proved that such a surface is a β -symplectic critical surface if and only if

$$\begin{cases} r f'(1 + (f')^2 + (g')^2)^{\frac{\beta-1}{2}} = C_1, \\ r g'(1 + (f')^2 + (g')^2)^{\frac{\beta-1}{2}} = C_2. \end{cases}$$

By analyzing the ODE system carefully, we give the following asymptotic expansions:

Theorem 1.5. *For any $\beta > 0$ and $\varepsilon > 0$, the equations*

$$\begin{cases} r f'(1 + (f')^2 + (g')^2)^{\frac{\beta-1}{2}} = 1, \\ r g'(1 + (f')^2 + (g')^2)^{\frac{\beta-1}{2}} = 1, \\ f(\varepsilon) = f_0, \\ g(\varepsilon) = g_0, \end{cases}$$

have a unique C^∞ -solution on $[\varepsilon, +\infty)$. Moreover, $f' = g'$. As $r \rightarrow \infty$, we have the asymptotic expansion,

$$f' = \frac{1}{r} - \frac{\beta-1}{r^3} + o(r^{-3}).$$

As $r \rightarrow 0$, we have the asymptotic expansion,

$$f' = 2^{\frac{1-\beta}{2\beta}} r^{-\frac{1}{\beta}} - \frac{\beta-1}{\beta} 2^{-\frac{3\beta+1}{2\beta}} r^{\frac{1}{\beta}} + o(r^{\frac{1}{\beta}}).$$

Furthermore, when $\beta \rightarrow 0$, we show that Σ_β converges to the catenoid locally, while when $\beta \rightarrow \infty$, Σ_β converges to a flat plane locally. This coincides with our expectation.

The following sections are organized as follows: in Section 2, we derive the Euler-Lagrange equation for the L_β functional and the elliptic equation satisfied by the Kähler angle, we also derive Webster's formula for β -symplectic critical surfaces; in Section 3, we prove that the equation is an elliptic system module tangential diffeomorphisms; in Section 4, we compute the second variation for L_β functional; in Section 5, we prove the openness of the set of β where there is a stable β -symplectic critical surface; in the last section, we study the rotationally symmetric β -symplectic critical surfaces in \mathbb{C}^2 .

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2. THE EULER-LAGRANGE EQUATION

Assume that $\phi_t : \Sigma \rightarrow M$ is a one-parameter family of immersions and $\frac{\partial \phi_t}{\partial t} |_{t=0} = \mathbf{X}$, \mathbf{X} is the variational vector field of Σ . We denote by $\bar{\nabla}$ the covariant derivative and by K the Riemannian curvature tensor on M . Furthermore, ∇, R denote the covariant derivative and the Riemannian curvature tensor of the induced metric g on the surface Σ .

Theorem 2.1. *Let M be a Kähler surface. The first variational formula of the functional L_β is, for any smooth normal vector field \mathbf{X} on Σ ,*

$$(2.1) \quad \delta_X L_\beta = -(\beta + 1) \int_\Sigma \frac{\mathbf{X} \cdot \mathbf{H}}{\cos^\beta \alpha} d\mu + \beta(\beta + 1) \int_\Sigma \frac{\mathbf{X} \cdot (J(J\nabla \cos \alpha)^\top)^\perp}{\cos^{\beta+3} \alpha} d\mu,$$

where \mathbf{H} is the mean curvature vector of Σ in M , and $()^\top$ means tangential components of $()$, $()^\perp$ means the normal components of $()$. The Euler-Lagrange equation of the functional L_β for $\beta \neq -1$ is

$$(2.2) \quad \cos^3 \alpha \mathbf{H} - \beta(J(J\nabla \cos \alpha)^\top)^\perp = 0.$$

Proof: Let $\{x_i\}$ be the local normal coordinates around a fixed point p on Σ . The induced metric on $\phi_t(\Sigma)$ is

$$g_{ij}(t) = \left\langle \frac{\partial \phi_t}{\partial x_i}, \frac{\partial \phi_t}{\partial x_j} \right\rangle.$$

For simplicity, we denote $\frac{\partial \phi_0}{\partial x_i}$ by e_i and $g_{ij}(t)$ by g_{ij} . A direct calculation gives

$$(2.3) \quad \frac{\partial}{\partial t} \Big|_{t=0} g_{ij} = \langle \bar{\nabla}_{e_i} \mathbf{X}, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} \mathbf{X} \rangle.$$

From the definition of Kähler angle we have

$$\cos \alpha_t = \frac{\omega(\partial\phi_t/\partial x^1, \partial\phi_t/\partial x^2)}{\sqrt{\det(g_t)}},$$

where $\det(g_t)$ is the determinant of the metric (g_t) . Now the functional can be written as

$$L_\beta(\phi_t) = \int_\Sigma \frac{\det^{(\beta+1)/2}(g_t)}{\omega^\beta(\partial\phi_t/\partial x^1, \partial\phi_t/\partial x^2)} dx^1 \wedge dx^2.$$

Thus,

$$(2.4) \quad \begin{aligned} & \frac{d}{dt} \Big|_{t=0} L_\beta(\phi_t) \\ &= (\beta+1) \int_\Sigma \frac{\langle \bar{\nabla}_{e_i} \mathbf{X}, e_i \rangle}{\cos^\beta \alpha} d\mu - \beta \int_\Sigma \frac{\omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X})}{\cos^{\beta+1} \alpha} d\mu. \end{aligned}$$

Now suppose \mathbf{X} is a normal vector field, then (2.4) can be written as

$$(2.5) \quad \begin{aligned} & \frac{d}{dt} \Big|_{t=0} L_\beta(\phi_t) \\ &= \int_\Sigma \left(\frac{\beta+1}{2} \frac{\partial_t g_{ij}|_{t=0} g^{ij}}{\cos^\beta \alpha} - \beta \frac{\partial_t \omega(\partial\phi_t/\partial x^1, \partial\phi_t/\partial x^2)|_{t=0}}{\cos^{\beta+1} \alpha} \right) d\mu \\ &= -(\beta+1) \int_\Sigma \frac{\mathbf{X} \cdot \mathbf{H}}{\cos^\beta \alpha} d\mu - \beta \int_\Sigma \frac{\omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X})}{\cos^{\beta+1} \alpha} d\mu. \end{aligned}$$

Since Σ is closed, applying the Stokes formula, we obtain

$$\begin{aligned} & -\beta \int_\Sigma \frac{\omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X})}{\cos^{\beta+1} \alpha} d\mu \\ &= -\beta \int_\Sigma \frac{e_1(\omega(\mathbf{X}, e_2)) - \omega(\mathbf{X}, \bar{\nabla}_{e_1} \bar{\nabla}_{e_2} F)}{\cos^{\beta+1} \alpha} d\mu \\ & \quad -\beta \int_\Sigma \frac{e_2(\omega(e_1, \mathbf{X})) - \omega(\bar{\nabla}_{e_2} \bar{\nabla}_{e_1} F, \mathbf{X})}{\cos^{\beta+1} \alpha} d\mu \\ &= -\beta \int_\Sigma \frac{\nabla_{e_1}(\omega(\mathbf{X}, e_2)) + \nabla_{e_2}(\omega(e_1, \mathbf{X}))}{\cos^{\beta+1} \alpha} d\mu \\ &= -\beta(\beta+1) \int_\Sigma \frac{\omega(\mathbf{X}, e_2) \nabla_{e_1} \cos \alpha + \omega(e_1, \mathbf{X}) \nabla_{e_2} \cos \alpha}{\cos^{\beta+2} \alpha} d\mu, \end{aligned}$$

where we have used the fact that ω is parallel. Since $\omega(\mathbf{X}, e_2) = -\langle \mathbf{X}, J e_2 \rangle$, $\omega(e_1, \mathbf{X}) = \langle \mathbf{X}, J e_1 \rangle$ and

$$\begin{aligned} (J \nabla \cos \alpha)^\top &= (J e_1 \nabla_{e_1} \cos \alpha + J e_2 \nabla_{e_2} \cos \alpha)^\top \\ &= \langle J e_1, e_2 \rangle e_2 \nabla_{e_1} \cos \alpha + \langle J e_2, e_1 \rangle e_1 \nabla_{e_2} \cos \alpha \\ &= (e_2 \nabla_{e_1} \cos \alpha - e_1 \nabla_{e_2} \cos \alpha) \cos \alpha, \end{aligned}$$

so

$$\omega(X, e_2) \nabla_{e_1} \cos \alpha + \omega(e_1, X) \nabla_{e_2} \cos \alpha = -\frac{\mathbf{X} \cdot (J(J \nabla \cos \alpha)^\top)^\perp}{\cos \alpha}.$$

Therefore, we have

$$\begin{aligned} \frac{d}{dt} \Big|_{t=0} L_\beta(\phi_t) &= -(\beta + 1) \int_\Sigma \frac{\mathbf{X} \cdot \mathbf{H}}{\cos^\beta \alpha} d\mu \\ &\quad + \beta(\beta + 1) \int_\Sigma \frac{\mathbf{X} \cdot (J(J \nabla \cos \alpha)^\top)^\perp}{\cos^{\beta+3} \alpha} d\mu. \end{aligned}$$

Q. E. D.

We express $(J(J \nabla \cos \alpha)^\top)^\perp$ at a fixed point p in a local frame. If we assume the Kähler form is self-dual, then J has the form

$$(2.6) \quad J = \begin{pmatrix} 0 & x & y & z \\ -x & 0 & z & -y \\ -y & -z & 0 & x \\ -z & y & -x & 0 \end{pmatrix},$$

where $x^2 + y^2 + z^2 = 1$. By the definition of the Kähler angle, we know that

$$x = \cos \alpha = \omega(e_1, e_2) = \langle J e_1, e_2 \rangle.$$

Then,

$$\begin{aligned} (J(J \nabla \cos \alpha)^\top)^\perp &= (J(\cos \alpha \partial_1 \cos \alpha e_2 - \cos \alpha \partial_2 \cos \alpha e_1))^\perp \\ &= -\cos \alpha \sin \alpha \partial_1 \alpha (J e_2)^\perp + \cos \alpha \sin \alpha \partial_2 \alpha (J e_1)^\perp \\ &= -\cos \alpha \sin \alpha \partial_1 \alpha (z e_3 - y e_4) + \cos \alpha \sin \alpha \partial_2 \alpha (y e_3 + z e_4). \end{aligned}$$

Thus, by (2.2) we get that

$$(2.7) \quad H^3 = \beta \frac{\sin \alpha}{\cos^2 \alpha} (y \partial_2 \alpha - z \partial_1 \alpha);$$

$$(2.8) \quad H^4 = \beta \frac{\sin \alpha}{\cos^2 \alpha} (y \partial_1 \alpha + z \partial_2 \alpha).$$

Theorem 2.2 ((cf. [4])). *If Σ is a closed symplectic surface which is smoothly immersed in M with the Kähler angle α , then α satisfies the following equation ,*

$$(2.9) \quad \begin{aligned} \Delta \cos \alpha &= \cos \alpha (-|h_{1k}^3 - h_{2k}^4|^2 - |h_{1k}^4 + h_{2k}^3|^2) \\ &\quad + \sin \alpha (H_{,1}^4 + H_{,2}^3) - \frac{\sin^2 \alpha}{\cos \alpha} (K_{1212} + K_{1234}). \end{aligned}$$

where K is the curvature operator of M and $H_{,i}^\alpha = \langle \bar{\nabla}_{e_i}^N \mathbf{H}, e_\alpha \rangle$.

Theorem 2.3. *Suppose that M is Kähler surface and Σ is a β -symplectic critical surface in M with Kähler angle α , then $\cos \alpha$ satisfies,*

$$\Delta \cos \alpha = \frac{2\beta \sin^2 \alpha}{\cos \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)} |\nabla \alpha|^2 - 2 \cos \alpha |\nabla \alpha|^2$$

$$(2.10) \quad -\frac{\cos^2 \alpha \sin^2 \alpha}{\cos^2 \alpha + \beta \sin^2 \alpha} Ric(Je_1, e_2).$$

Proof. We will compute pointwise. For a fixed point $p \in \Sigma$, we can choose the local frame such that at p , $y = \sin \alpha$ and $z = 0$. For a β -symplectic critical surface Σ , if we set $\mathbf{V} = \nabla_{e_2} \alpha e_3 + \nabla_{e_1} \alpha e_4$, then we have

$$(2.11) \quad \mathbf{H} = \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} \mathbf{V}.$$

It is easy to check that (see (2.5) of [4])

$$(2.12) \quad \nabla_{e_1} \alpha = -(h_{11}^4 + h_{12}^3), \quad \nabla_{e_2} \alpha = -(h_{12}^4 + h_{22}^3).$$

By direct computation, we have at p ,

$$\begin{aligned} (h_{1k}^3 - h_{2k}^4)^2 + (h_{1k}^4 + h_{2k}^3)^2 &= |\mathbf{H}|^2 + 2|\mathbf{V}|^2 + 2\mathbf{H} \cdot \mathbf{V} \\ &= \left(\beta^2 \frac{\sin^4 \alpha}{\cos^4 \alpha} + 2 + 2\beta \frac{\sin^2 \alpha}{\cos^2 \alpha}\right) |\mathbf{V}|^2 \\ &= \frac{\beta^2 \sin^4 \alpha + 2 \cos^4 \alpha + 2\beta \sin^2 \alpha \cos^2 \alpha}{\cos^4 \alpha} |\nabla \alpha|^2, \end{aligned}$$

and

$$\begin{aligned} \sin \alpha (H_{,1}^4 + H_{,2}^3) &= \sin \alpha (\langle \bar{\nabla}_{e_1} \mathbf{H}, e_4 \rangle + \langle \bar{\nabla}_{e_2} \mathbf{H}, e_3 \rangle) \\ &= \sin \alpha (\partial_1(H^4) + H^3 \langle \bar{\nabla}_{e_1} e_3, e_4 \rangle + \partial_2(H^3) + H^4 \langle \bar{\nabla}_{e_2} e_4, e_3 \rangle) \\ &= \beta \sin \alpha \left\{ \partial_1 \left[\frac{\sin \alpha}{\cos^2 \alpha} (y \partial_1 \alpha + z \partial_2 \alpha) \right] + \partial_2 \left[\frac{\sin \alpha}{\cos^2 \alpha} (y \partial_2 \alpha - z \partial_1 \alpha) \right] \right\} \\ &\quad + \sin \alpha (H^3 \langle \bar{\nabla}_{e_1} e_3, e_4 \rangle + H^4 \langle \bar{\nabla}_{e_2} e_4, e_3 \rangle) \\ &= \beta \sin \alpha \left[\partial_1 \left(\frac{\sin \alpha}{\cos^2 \alpha} \partial_1 \alpha \right) y + \frac{\sin \alpha}{\cos^2 \alpha} \partial_1 \alpha \partial_1 y \right. \\ &\quad + \partial_1 \left(\frac{\sin \alpha}{\cos^2 \alpha} \partial_2 \alpha \right) z + \frac{\sin \alpha}{\cos^2 \alpha} \partial_2 \alpha \partial_1 z \\ &\quad + \partial_2 \left(\frac{\sin \alpha}{\cos^2 \alpha} \partial_2 \alpha \right) y + \frac{\sin \alpha}{\cos^2 \alpha} \partial_2 \alpha \partial_2 y \\ &\quad \left. - \partial_2 \left(\frac{\sin \alpha}{\cos^2 \alpha} \partial_1 \alpha \right) z - \frac{\sin \alpha}{\cos^2 \alpha} \partial_1 \alpha \partial_2 z \right] \\ &\quad + \sin \alpha (H^3 \langle \bar{\nabla}_{e_1} e_3, e_4 \rangle + H^4 \langle \bar{\nabla}_{e_2} e_4, e_3 \rangle) \\ &= \beta \sin^2 \alpha \left[\partial_1 \left(\frac{\sin \alpha}{\cos^2 \alpha} \partial_1 \alpha \right) + \partial_2 \left(\frac{\sin \alpha}{\cos^2 \alpha} \partial_2 \alpha \right) \right] \\ &\quad + \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} (\partial_1 \alpha \partial_1 y + \partial_2 \alpha \partial_2 y) + \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} (\partial_2 \alpha \partial_1 z - \partial_1 \alpha \partial_2 z) \\ &\quad + \sin \alpha (H^3 \langle \bar{\nabla}_{e_1} e_3, e_4 \rangle + H^4 \langle \bar{\nabla}_{e_2} e_4, e_3 \rangle) \\ &= \beta \frac{\sin^3 \alpha}{\cos^2 \alpha} \Delta \alpha + \beta \frac{\sin^2 \alpha (1 + \sin^2 \alpha)}{\cos^3 \alpha} |\nabla \alpha|^2 \\ &\quad + \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} (\partial_1 \alpha \partial_1 y + \partial_2 \alpha \partial_2 y) + \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} (\partial_2 \alpha \partial_1 z - \partial_1 \alpha \partial_2 z) \end{aligned}$$

$$(2.13) \quad +\beta \frac{\sin^3 \alpha}{\cos^2 \alpha} (\partial_2 \alpha \langle \bar{\nabla}_{e_1} e_3, e_4 \rangle + \partial_1 \alpha \langle \bar{\nabla}_{e_2} e_4, e_3 \rangle).$$

Now we begin to compute $\partial_1 \alpha \partial_1 y + \partial_2 \alpha \partial_2 y$ and $\partial_2 \alpha \partial_1 z - \partial_1 \alpha \partial_2 z$. Note that $y = \langle J e_1, e_3 \rangle$. Then

$$\begin{aligned} \partial_1 \alpha \partial_1 y + \partial_2 \alpha \partial_2 y &= \partial_1 \alpha (\langle J \bar{\nabla}_{e_1} e_1, e_3 \rangle + \langle J e_1, \bar{\nabla}_{e_1} e_3 \rangle) \\ &\quad + \partial_2 \alpha (\langle J \bar{\nabla}_{e_2} e_1, e_3 \rangle + \langle J e_1, \bar{\nabla}_{e_2} e_3 \rangle) \\ &= \partial_1 \alpha (-\cos \alpha h_{11}^4 - \cos \alpha h_{12}^3) + \partial_2 \alpha (-\cos \alpha h_{12}^4 - \cos \alpha h_{22}^3) \\ &= \cos \alpha (-h_{11}^4 - h_{12}^3) \partial_1 \alpha + \cos \alpha (-h_{12}^4 - h_{22}^3) \partial_2 \alpha \\ &= \cos \alpha |\nabla \alpha|^2. \end{aligned}$$

Similarly, since $z = \langle J e_1, e_4 \rangle$, we have

$$\begin{aligned} \partial_2 \alpha \partial_1 z - \partial_1 \alpha \partial_2 z &= \partial_2 \alpha (\langle J \bar{\nabla}_{e_1} e_1, e_4 \rangle + \langle J e_1, \bar{\nabla}_{e_1} e_4 \rangle) \\ &\quad - \partial_1 \alpha (\langle J \bar{\nabla}_{e_2} e_1, e_4 \rangle + \langle J e_1, \bar{\nabla}_{e_2} e_4 \rangle) \\ &= \partial_2 \alpha (-\cos \alpha h_{12}^4 + \sin \alpha \langle \bar{\nabla}_{e_1} e_4, e_3 \rangle + \cos \alpha h_{11}^3) \\ &\quad - \partial_1 \alpha (-\cos \alpha h_{22}^4 + \sin \alpha \langle \bar{\nabla}_{e_2} e_4, e_3 \rangle + \cos \alpha h_{12}^3) \\ &= \partial_2 \alpha (H^3 \cos \alpha + \cos \alpha \partial_2 \alpha) + \partial_1 \alpha (H^4 \cos \alpha + \cos \alpha \partial_1 \alpha) \\ &\quad + \sin \alpha (\partial_2 \alpha \langle \bar{\nabla}_{e_1} e_4, e_3 \rangle - \partial_1 \alpha \langle \bar{\nabla}_{e_2} e_4, e_3 \rangle) \\ &= \frac{\cos^2 \alpha + \beta \sin^2 \alpha}{\cos \alpha} |\nabla \alpha|^2 + \sin \alpha (\partial_2 \alpha \langle \bar{\nabla}_{e_1} e_4, e_3 \rangle - \partial_1 \alpha \langle \bar{\nabla}_{e_2} e_4, e_3 \rangle). \end{aligned}$$

Putting this equation into (2.13) we get that

$$(2.14) \quad \sin \alpha (H_{,1}^4 + H_{,2}^3) = -\beta \frac{\sin^2 \alpha}{\cos^2 \alpha} \Delta \cos \alpha + \beta \frac{\sin^2 \alpha (2 + \beta \sin^2 \alpha)}{\cos^3 \alpha} |\nabla \alpha|^2.$$

Putting these two equations into (2.9), we obtain that,

$$\begin{aligned} \Delta \cos \alpha &= -\frac{\beta^2 \sin^4 \alpha + 2 \cos^4 \alpha + 2 \beta \sin^2 \alpha \cos^2 \alpha}{\cos^3 \alpha} |\nabla \alpha|^2 - \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} \Delta \cos \alpha \\ &\quad + \beta \frac{\sin^2 \alpha (2 + \beta \sin^2 \alpha)}{\cos^3 \alpha} |\nabla \alpha|^2 - \frac{\sin^2 \alpha}{\cos \alpha} (K_{1212} + K_{1234}) \\ &= \frac{2 \beta \sin^2 \alpha - 2 \cos^2 \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^3 \alpha} |\nabla \alpha|^2 \\ &\quad - \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} \Delta \cos \alpha - \frac{\sin^2 \alpha}{\cos \alpha} (K_{1212} + K_{1234}). \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta \cos \alpha &= \frac{2 \beta \sin^2 \alpha}{\cos \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)} |\nabla \alpha|^2 - 2 \cos \alpha |\nabla \alpha|^2 \\ &\quad - \frac{\cos \alpha \sin^2 \alpha}{\cos^2 \alpha + \beta \sin^2 \alpha} (K_{1212} + K_{1234}) \end{aligned}$$

$$\begin{aligned}
&= \frac{2\beta \sin^2 \alpha}{\cos \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)} |\nabla \alpha|^2 - 2 \cos \alpha |\nabla \alpha|^2 \\
&\quad - \frac{\cos^2 \alpha \sin^2 \alpha}{\cos^2 \alpha + \beta \sin^2 \alpha} \text{Ric}(Je_1, e_2).
\end{aligned}$$

The last equality used the fact that $K_{1212} + K_{1234} = \cos \alpha \text{Ric}(Je_1, e_2)$ (Lemma 3.2 in [4]). Q. E. D.

Corollary 2.4. *Assume M is Kähler-Einstein surface with scalar curvature K , then $\cos \alpha$ satisfies,*

$$\Delta \cos \alpha = \frac{2\beta \sin^2 \alpha}{\cos \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)} |\nabla \alpha|^2 - 2 \cos \alpha |\nabla \alpha|^2 - \frac{K}{4} \frac{\cos^3 \alpha \sin^2 \alpha}{\cos^2 \alpha + \beta \sin^2 \alpha}.$$

Corollary 2.5. *Any β -symplectic critical surface in a Kähler-Einstein surface with nonnegative scalar curvature is a holomorphic curve for $\beta \geq 0$.*

Similar to minimal surfaces ([9], [10]) and symplectic critical surfaces ([4]), a non-holomorphic β -symplectic critical surface in a Kähler surface has at most finite complex points, and the following formula can be derived.

Theorem 2.6. *Suppose that Σ is a non holomorphic β -symplectic critical surface in a Kähler surface M . Then*

$$\chi(\Sigma) + \chi(\nu) = -P,$$

and

$$c_1(M)([\Sigma]) = -P,$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ , $\chi(\nu)$ is the Euler characteristic of the normal bundle of Σ in M , $c_1(M)$ is the first Chern class of M , $[\Sigma] \in H_2(M, \mathbf{Z})$ is the homology class of Σ in M , and P is the number of complex tangent points.

Proof : Similar to the analysis in [4], we know that the complex points are isolated. Set

$$g(\alpha) = \ln(\sin^2 \alpha).$$

Then using the equation (2.10), we obtain

$$\begin{aligned}
\Delta g(\alpha) &= -2|\nabla \alpha|^2 - 2 \frac{\cos \alpha}{\sin^2 \alpha} \Delta \cos \alpha - 4 \frac{\cos^2 \alpha}{\sin^2 \alpha} |\nabla \alpha|^2 \\
&= -2|\nabla \alpha|^2 - \frac{4\beta}{\cos^2 \alpha + \beta \sin^2 \alpha} |\nabla \alpha|^2 \\
&\quad + \frac{2 \cos^2 \alpha}{\cos^2 \alpha + \beta \sin^2 \alpha} (K_{1212} + K_{1234}).
\end{aligned} \tag{2.15}$$

This equation is valid away from the complex tangent points of M . By the Gauss equation and Ricci equation, we have,

$$\begin{aligned}
R_{1212} &= K_{1212} + h_{11}^\alpha h_{22}^\alpha - (h_{12}^\alpha)^2 \\
R_{1234} &= K_{1234} + h_{1k}^3 h_{2k}^4 - h_{1k}^4 h_{2k}^3,
\end{aligned}$$

where R_{1212} is the curvature of $T\Sigma$ and R_{1234} is the curvature of the normal bundle ν . Adding these two equations together, we get that,

$$\begin{aligned} K_{1212} + K_{1234} &= R_{1212} + R_{1234} - \frac{1}{2}|\mathbf{H}|^2 \\ &\quad + \frac{1}{2}((h_{1k}^3 - h_{2k}^4)^2 + (h_{1k}^4 + h_{2k}^3)^2) \\ &= R_{1212} + R_{1234} + |\mathbf{V}|^2 + \mathbf{H} \cdot \mathbf{V} \\ &= R_{1212} + R_{1234} + \frac{\cos^2 \alpha + \beta \sin^2 \alpha}{\cos^2 \alpha} |\nabla \alpha|^2. \end{aligned}$$

Thus,

$$R_{1212} + R_{1234} = \frac{\cos^2 \alpha + \beta \sin^2 \alpha}{2 \cos^2 \alpha} \Delta g(\alpha) + \frac{2\beta}{\cos^2 \alpha} |\nabla \alpha|^2.$$

Integrating the above equality over Σ , arguing as that in [9], we can obtain,

$$(2.16) \quad 2\pi(\chi(T\Sigma) + \chi(\nu)) = -2\pi P,$$

where $\chi(T\Sigma)$ is the Euler characteristic of Σ and $\chi(\nu)$ is the Euler characteristic of the normal bundle of Σ in M , P is the sum of the orders of complex tangent points.

By (2.15) we also get that,

$$\begin{aligned} \Delta g(\alpha) &= -2|\nabla \alpha|^2 - \frac{4\beta}{\cos^2 \alpha + \beta \sin^2 \alpha} |\nabla \alpha|^2 \\ &\quad + \frac{2 \cos^3 \alpha}{\cos^2 \alpha + \beta \sin^2 \alpha} Ric(Je_1, e_2). \end{aligned}$$

Note that $Ric(Je_1, e_2)d\mu_\Sigma$ is the pulled back Ricci 2-form of M by the immersion F to Σ , i.e,

$$F^*(Ric^M) = Ric(Je_1, e_2)d\mu_\Sigma.$$

Thus,

$$F^*(Ric^M) = \left(\frac{\cos^2 \alpha + \beta \sin^2 \alpha}{2 \cos^3 \alpha} \Delta g(\alpha) + \frac{2\beta + \cos^2 \alpha + \beta \sin^2 \alpha}{\cos^3 \alpha} |\nabla \alpha|^2 \right) d\mu_\Sigma.$$

Integrating it over Σ , we obtain that,

$$(2.17) \quad 2\pi F^* c_1(M)[\Sigma] = -2\pi P.$$

Q.E.D.

Remark 2.7. In [4] there was an error in the computations which was corrected above.

Remark 2.8. The formulae (2.16) and (2.17) are usually called Webster's formula, and proved by Webster in [6] and [7].

3. PRINCIPAL SYMBOL OF THE EQUATION (2.2)

In this section, we examine the principle symbol of the equation (2.2). First note that, the equation (2.2) can be rewritten as

$$(3.1) \quad \cos^2 \alpha \mathbf{H} = \beta(J(\nabla_{e_1} \cos \alpha e_2 - \nabla_{e_2} \cos \alpha e_1))^\perp.$$

For simplicity, we suppose that Σ is a surface in \mathbb{C}^2 . The general case is similar. In local coordinate, we can express the surface as

$$\begin{aligned} F : \Sigma &\longrightarrow \mathbb{C}^2 = \mathbb{R}^4 \\ (x_1, x_2) &\longmapsto F(x_1, x_2) = (F^1(x_1, x_2), \dots, F^4(x_1, x_2)). \end{aligned}$$

We will use the following conventions:

$$1 \leq i, j, \dots \leq 2, \quad 3 \leq \alpha, \beta, \dots \leq 4, \quad 1 \leq A, B, \dots \leq 4.$$

The tangent space of Σ at a fixed point $x \in \Sigma$ is spanned by $\{e_1, e_2\}$ given by

$$(3.2) \quad e_1 = \frac{\partial F}{\partial x_1} = \frac{\partial F^A}{\partial x_1} E_A, \quad e_2 = \frac{\partial F}{\partial x_2} = \frac{\partial F^A}{\partial x_2} E_A,$$

where $\{E_1, \dots, E_4\}$ is the standard orthonormal basis of \mathbb{R}^4 . Therefore, the induced metric on Σ is given by

$$(3.3) \quad g_{ij} = \langle e_i, e_j \rangle = \frac{\partial F^A}{\partial x_i} \frac{\partial F^A}{\partial x_j}.$$

We can take the coordinates so that at the fixed point $x \in \Sigma$, we have $g_{ij}(x) = \delta_{ij}$. We will also take the standard complex structure J on \mathbb{C}^2 given by

$$(3.4) \quad J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then we have

$$(3.5) \quad \begin{cases} J E_{2k-1} = E_{2k}, \\ J E_{2k} = -E_{2k-1}. \end{cases}$$

Furthermore, we choose any orthonormal basis $\{e_\alpha\}_{\alpha=3}^4$ of the normal space. Denote

$$(3.6) \quad P = \cos^2 \alpha \mathbf{H} - \beta(J(\nabla_{e_1} \cos \alpha e_2 - \nabla_{e_2} \cos \alpha e_1))^\perp.$$

We will compute the principal symbol of P .

First we consider the principal part of \mathbf{H} . Note that by (3.3), we can easily see that the Christoffel symbol of the induced metric is:

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} \left\{ \frac{\partial g_{il}}{\partial x_j} + \frac{\partial g_{jl}}{\partial x_i} - \frac{\partial g_{ij}}{\partial x_l} \right\} = g^{kl} \frac{\partial^2 F^B}{\partial x_i \partial x_j} \frac{\partial F^B}{\partial x_l}.$$

Therefore, we have

$$\begin{aligned}
 \mathbf{H} = \Delta_\Sigma F &= g^{ij} \left(\frac{\partial^2 F}{\partial x_i \partial x_j} - \Gamma_{ij}^k \frac{\partial F}{\partial x_k} \right) \\
 &= g^{ij} \left(\frac{\partial^2 F}{\partial x_i \partial x_j} - g^{kl} \frac{\partial^2 F^B}{\partial x_i \partial x_j} \frac{\partial F^B}{\partial x_l} \frac{\partial F}{\partial x_k} \right) \\
 (3.7) \quad &= g^{ij} \left(\frac{\partial^2 F^A}{\partial x_i \partial x_j} - g^{kl} \frac{\partial F^A}{\partial x_k} \frac{\partial F^B}{\partial x_l} \frac{\partial^2 F^B}{\partial x_i \partial x_j} \right) E_A.
 \end{aligned}$$

The linearization of the operator at F in the direction G is:

$$(3.8) \quad D(\mathbf{H})(F)G = g^{ij} \left(\frac{\partial^2 G^A}{\partial x_i \partial x_j} - g^{kl} \frac{\partial F^A}{\partial x_k} \frac{\partial F^B}{\partial x_l} \frac{\partial^2 G^B}{\partial x_i \partial x_j} \right) E_A + \text{first order terms}.$$

Next, we will consider the second part of P . By definition,

$$(3.9) \quad \cos \alpha = \frac{\omega(e_1, e_2)}{\sqrt{\det(g_{ij})}} = \frac{\langle J e_1, e_2 \rangle}{\sqrt{\det(g_{ij})}}.$$

By (3.2) and (3.4), we have:

$$\langle J e_1, e_2 \rangle = \sum_{k=1}^2 \left(\frac{\partial F^{2k-1}}{\partial x_1} \frac{\partial F^{2k}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1} \frac{\partial F^{2k-1}}{\partial x_2} \right).$$

Therefore, we have

$$\cos \alpha = \frac{\sum_{k=1}^2 \left(\frac{\partial F^{2k-1}}{\partial x_1} \frac{\partial F^{2k}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1} \frac{\partial F^{2k-1}}{\partial x_2} \right)}{\sqrt{\det(g_{ij})}},$$

$$\begin{aligned}
 &\frac{\partial \cos \alpha}{\partial x_1} = \\
 &= \frac{1}{\sqrt{\det(g_{ij})}} \left\{ \sum_{k=1}^2 \left(\frac{\partial^2 F^{2k-1}}{\partial x_1^2} \frac{\partial F^{2k}}{\partial x_2} + \frac{\partial F^{2k-1}}{\partial x_1} \frac{\partial^2 F^{2k}}{\partial x_1 \partial x_2} - \frac{\partial^2 F^{2k}}{\partial x_1^2} \frac{\partial F^{2k-1}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1} \frac{\partial^2 F^{2k-1}}{\partial x_1 \partial x_2} \right) \right. \\
 &\quad \left. - \frac{1}{2} \langle J e_1, e_2 \rangle g^{ij} \left(\frac{\partial^2 F^A}{\partial x_1 \partial x_i} \frac{\partial F^A}{\partial x_j} + \frac{\partial F^A}{\partial x_i} \frac{\partial^2 F^A}{\partial x_1 \partial x_j} \right) \right\},
 \end{aligned}$$

and

$$\begin{aligned}
 &\frac{\partial \cos \alpha}{\partial x_2} = \\
 &= \frac{1}{\sqrt{\det(g_{ij})}} \left\{ \sum_{k=1}^2 \left(\frac{\partial^2 F^{2k-1}}{\partial x_1 \partial x_2} \frac{\partial F^{2k}}{\partial x_2} + \frac{\partial F^{2k-1}}{\partial x_1} \frac{\partial^2 F^{2k}}{\partial x_2^2} - \frac{\partial^2 F^{2k}}{\partial x_1 \partial x_2} \frac{\partial F^{2k-1}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1} \frac{\partial^2 F^{2k-1}}{\partial x_2^2} \right) \right. \\
 &\quad \left. - \frac{1}{2} \langle J e_1, e_2 \rangle g^{ij} \left(\frac{\partial^2 F^A}{\partial x_2 \partial x_i} \frac{\partial F^A}{\partial x_j} + \frac{\partial F^A}{\partial x_i} \frac{\partial^2 F^A}{\partial x_2 \partial x_j} \right) \right\}.
 \end{aligned}$$

By our choice of the frame, at the fixed point x , we have

$$\begin{aligned}
 & (J(\nabla_{e_1} \cos \alpha e_2 - \nabla_{e_2} \cos \alpha e_1))^\perp \\
 &= \frac{\partial \cos \alpha}{\partial x_1} (Je_2)^\perp - \frac{\partial \cos \alpha}{\partial x_2} (Je_1)^\perp \\
 (3.10) \quad &= \frac{\partial \cos \alpha}{\partial x_1} \langle Je_2, e_\alpha \rangle e_\alpha - \frac{\partial \cos \alpha}{\partial x_2} \langle Je_1, e_\alpha \rangle e_\alpha.
 \end{aligned}$$

Notice that $\cos \alpha$, g_{ij} , e_i and e_α only involve first order derivatives of the immersion F . Therefore, by (3.8) and (3.10), we know that the linearization of the operator P at F in the direction G (computed at the point x) is:

$$\begin{aligned}
 & D(\mathbf{P})(F)G \\
 &= \cos^2 \alpha g^{ij} \left(\frac{\partial^2 G^A}{\partial x_i \partial x_j} - g^{kl} \frac{\partial F^A}{\partial x_k} \frac{\partial F^B}{\partial x_l} \frac{\partial^2 G^B}{\partial x_i \partial x_j} \right) E_A \\
 & - \beta \left\{ \sum_{k=1}^n \left(\frac{\partial^2 G^{2k-1}}{\partial x_1^2} \frac{\partial F^{2k}}{\partial x_2} + \frac{\partial F^{2k-1}}{\partial x_1} \frac{\partial^2 G^{2k}}{\partial x_1 \partial x_2} - \frac{\partial^2 G^{2k}}{\partial x_1^2} \frac{\partial F^{2k-1}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1} \frac{\partial^2 G^{2k-1}}{\partial x_1 \partial x_2} \right) \right. \\
 & \quad \left. - \frac{1}{2} \langle Je_1, e_2 \rangle g^{ij} \left(\frac{\partial^2 G^A}{\partial x_1 \partial x_i} \frac{\partial F^A}{\partial x_j} + \frac{\partial F^A}{\partial x_i} \frac{\partial^2 G^A}{\partial x_1 \partial x_j} \right) \right\} \langle Je_2, e_\alpha \rangle e_\alpha \\
 & + \beta \left\{ \sum_{k=1}^n \left(\frac{\partial^2 G^{2k-1}}{\partial x_1 \partial x_2} \frac{\partial F^{2k}}{\partial x_2} + \frac{\partial F^{2k-1}}{\partial x_1} \frac{\partial^2 G^{2k}}{\partial x_2^2} - \frac{\partial^2 G^{2k}}{\partial x_1 \partial x_2} \frac{\partial F^{2k-1}}{\partial x_2} - \frac{\partial F^{2k}}{\partial x_1} \frac{\partial^2 G^{2k-1}}{\partial x_2^2} \right) \right. \\
 & \quad \left. - \frac{1}{2} \langle Je_1, e_2 \rangle g^{ij} \left(\frac{\partial^2 G^A}{\partial x_2 \partial x_i} \frac{\partial F^A}{\partial x_j} + \frac{\partial F^A}{\partial x_i} \frac{\partial^2 G^A}{\partial x_2 \partial x_j} \right) \right\} \langle Je_1, e_\alpha \rangle e_\alpha \\
 (3.11) & + \text{first order terms.}
 \end{aligned}$$

We will denote G^T and G^\perp the projection of $G \in \mathbb{R}^4$ on the tangent bundle and normal bundle of Σ respectively. It is easy to see that

$$|G^T|^2 = g^{kl} \langle G, \frac{\partial F}{\partial x_k} \rangle \langle G, \frac{\partial F}{\partial x_l} \rangle = g^{kl} G^A G^B \frac{\partial F^A}{\partial x_k} \frac{\partial F^B}{\partial x_l}.$$

Then we see that the principal symbol of P is given by:

$$\begin{aligned}
 & \langle \sigma(D(P))(x, \xi)G, G \rangle \\
 &= \cos^2 \alpha g^{ij} \left(\xi_i \xi_j |G|^2 - g^{kl} \frac{\partial F^A}{\partial x_k} \frac{\partial F^B}{\partial x_l} \xi_i \xi_j G^A G^B \right) \\
 & - \beta \left\{ \sum_{k=1}^n \left(G^{2k-1} \frac{\partial F^{2k}}{\partial x_2} - G^{2k} \frac{\partial F^{2k-1}}{\partial x_2} \right) \langle Je_2, G^\perp \rangle \xi_1^2 \right. \\
 & \quad \left. + \sum_{k=1}^n \left(G^{2k-1} \frac{\partial F^{2k}}{\partial x_1} - G^{2k} \frac{\partial F^{2k-1}}{\partial x_1} \right) \langle Je_1, G^\perp \rangle \xi_2^2 \right\}
 \end{aligned}$$

$$\begin{aligned}
& - \sum_{k=1}^n \left[\left(G^{2k-1} \frac{\partial F^{2k}}{\partial x_1} - G^{2k} \frac{\partial F^{2k-1}}{\partial x_1} \right) \langle J e_2, G^\perp \rangle \right. \\
& \quad \left. + \left(G^{2k-1} \frac{\partial F^{2k}}{\partial x_2} - G^{2k} \frac{\partial F^{2k-1}}{\partial x_2} \right) \langle J e_1, G^\perp \rangle \right] \xi_1 \xi_2 \\
(3.12) \quad & + \cos \alpha g^{ij} G^A \left(\frac{\partial F^A}{\partial x_j} \langle J e_1, G^\perp \rangle \xi_2 \xi_i - \frac{\partial F^A}{\partial x_j} \langle J e_2, G^\perp \rangle \xi_1 \xi_i \right) \Big\}.
\end{aligned}$$

By (3.5), we have

$$\begin{aligned}
& \langle \sigma(D(P))(x, \xi) G, G \rangle \\
& = \cos^2 \alpha |\xi|^2 |G^\perp|^2 \\
& \quad - \beta \{ (-\langle J e_2, G^\perp \rangle \langle J e_2, G \rangle - \langle J e_1, e_2 \rangle \langle G, e_1 \rangle \langle J e_2, G^\perp \rangle) \xi_1^2 \\
& \quad + (-\langle J e_1, G^\perp \rangle \langle J e_1, G \rangle + \langle J e_1, e_2 \rangle \langle G, e_2 \rangle \langle J e_1, G^\perp \rangle) \xi_2^2 \\
& \quad + (\langle J e_2, G^\perp \rangle \langle J e_1, G \rangle + \langle J e_1, G^\perp \rangle \langle J e_2, G \rangle \\
(3.13) \quad & + \langle J e_1, e_2 \rangle \langle G, e_1 \rangle \langle J e_1, G^\perp \rangle - \langle J e_1, e_2 \rangle \langle G, e_2 \rangle \langle J e_2, G^\perp \rangle) \xi_1 \xi_2 \}.
\end{aligned}$$

Note that $(J e_1)^T = \langle J e_1, e_2 \rangle e_2$ and $(J e_2)^T = -\langle J e_1, e_2 \rangle e_1$. Thus we have

$$\begin{aligned}
\langle \sigma(D(P))(x, \xi) G, G \rangle & = \cos^2 \alpha |\xi|^2 |G^\perp|^2 \\
& \quad + \beta (\langle G^\perp, J e_2 \rangle^2 \xi_1^2 - 2 \langle G^\perp, J e_1 \rangle \langle G^\perp, J e_2 \rangle \xi_1 \xi_2 + \langle G^\perp, J e_1 \rangle^2 \xi_2^2) \\
& = (\cos^2 \alpha |G^\perp|^2 + \beta \langle G^\perp, J e_2 \rangle^2) \xi_1^2 - 2 \beta \langle G^\perp, J e_1 \rangle \langle G^\perp, J e_2 \rangle \xi_1 \xi_2 \\
(3.14) \quad & + (\cos^2 \alpha |G^\perp|^2 + \beta \langle G^\perp, J e_1 \rangle^2) \xi_2^2.
\end{aligned}$$

The coefficient matrix is given by

$$(3.15) \quad O = \begin{pmatrix} \cos^2 \alpha |G^\perp|^2 + \beta \langle G^\perp, J e_2 \rangle^2 & -\beta \langle G^\perp, J e_1 \rangle \langle G^\perp, J e_2 \rangle \\ -\beta \langle G^\perp, J e_1 \rangle \langle G^\perp, J e_2 \rangle & \cos^2 \alpha |G^\perp|^2 + \beta \langle G^\perp, J e_1 \rangle^2 \end{pmatrix}.$$

We have

$$\begin{aligned}
\det O & = \cos^4 \alpha |G^\perp|^4 + \beta \cos^2 \alpha |G^\perp|^2 (\langle G^\perp, J e_1 \rangle^2 + \langle G^\perp, J e_2 \rangle^2) \\
(3.16) \quad & = \cos^2 \alpha |G^\perp|^2 [\cos^2 \alpha |G^\perp|^2 + \beta (\langle G^\perp, J e_1 \rangle^2 + \langle G^\perp, J e_2 \rangle^2)].
\end{aligned}$$

From (3.16), we see that if $\beta \geq 0$, then $\det O \geq 0$, with equality holds if and only if $\cos^2 \alpha |G^\perp|^2 = 0$. In particular,

Proposition 3.1. *For a β -symplectic critical surface with $\beta \geq 0$, the Euler-Lagrange equation (2.2) is an elliptic system modulo tangential diffeomorphisms of Σ .*

4. SECOND VARIATION FORMULA

In this section, we will compute the second variation of the functional L_β for a β -symplectic critical surface. To this end, we use the notations as in Section 2. Recall

that

$$L_\beta(\phi) = \int_\Sigma \frac{1}{\cos^\beta \alpha} d\mu,$$

where $\phi : \Sigma \rightarrow M$ is a smooth symplectic immersion. Let $\phi(\Sigma)$ be a β -symplectic critical surface. We consider a smooth family of symplectic immersions

$$\phi_{t,\epsilon} : \Sigma \times (-\delta, \delta) \times (-a, a) \rightarrow M$$

with $\phi_{0,0} = \phi$. Since symplectic is an open condition, we can assume that $\phi_{t,\epsilon}(\Sigma)$ is symplectic property for every $t \in (-\delta, \delta)$ and every $\epsilon \in (-a, a)$, that means that $\phi_{t,\epsilon}$ is a symplectic variation of ϕ . For simplicity, we write $\phi_{0,0}(\Sigma) = \phi(\Sigma) = \Sigma$. Let

$$\frac{\partial \phi_{t,0}}{\partial t} \big|_{t=0} = \mathbf{X}, \quad \frac{\partial \phi_{0,\epsilon}}{\partial \epsilon} \big|_{\epsilon=0} = \mathbf{Y}, \quad \text{and} \quad \frac{\partial^2 \phi_{t,\epsilon}}{\partial t \partial \epsilon} \big|_{t=0, \epsilon=0} = \mathbf{Z}.$$

Lemma 4.1. *We have the general second variation formula:*

$$\begin{aligned} & \frac{\partial^2}{\partial t \partial \epsilon} \big|_{t=0, \epsilon=0} L_\beta(\phi_{t,\epsilon}) \\ = & (\beta+1) \int_\Sigma \frac{\langle \bar{\nabla}_{e_i}^\perp \mathbf{X}, \bar{\nabla}_{e_i}^\perp \mathbf{Y} \rangle}{\cos^\beta \alpha} d\mu - (\beta+1) \int_\Sigma \frac{K(\mathbf{X}, e_i, \mathbf{Y}, e_i)}{\cos^\beta \alpha} d\mu \\ & + (\beta+1)^2 \int_\Sigma \frac{\text{div}_\Sigma \mathbf{X} \text{div}_\Sigma \mathbf{Y}}{\cos^\beta \alpha} d\mu \\ & - \frac{\beta+1}{2} \int_\Sigma \frac{\langle e_i, \bar{\nabla}_{e_j} \mathbf{X} \rangle \langle e_j, \bar{\nabla}_{e_i} \mathbf{Y} \rangle + \langle e_j, \bar{\nabla}_{e_i} \mathbf{X} \rangle \langle e_i, \bar{\nabla}_{e_j} \mathbf{Y} \rangle}{\cos^\beta \alpha} d\mu \\ & - \beta(\beta+1) \int_\Sigma \frac{\text{div}_\Sigma \mathbf{X} (\omega(\bar{\nabla}_{e_1} \mathbf{Y}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{Y}))}{\cos^{\beta+1} \alpha} d\mu \\ & - \beta(\beta+1) \int_\Sigma \frac{\text{div}_\Sigma \mathbf{Y} (\omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X}))}{\cos^{\beta+1} \alpha} d\mu \\ & - \beta \int_\Sigma \frac{\omega(K(\mathbf{Y}, e_1) \mathbf{X}, e_2) \omega(e_1, K(\mathbf{Y}, e_2) \mathbf{X})}{\cos^{\beta+1} \alpha} d\mu \\ (4.1) \quad & - \beta \int_\Sigma \frac{\omega(\bar{\nabla}_{e_1} \mathbf{X}, \bar{\nabla}_{e_2} \mathbf{Y}) + \omega(\bar{\nabla}_{e_1} \mathbf{Y}, \bar{\nabla}_{e_2} \mathbf{X})}{\cos^{\beta+1} \alpha} d\mu \\ & + \beta(\beta+1) \int_\Sigma \frac{(\omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X})) (\omega(\bar{\nabla}_{e_1} \mathbf{Y}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{Y}))}{\cos^{\beta+2} \alpha} d\mu, \end{aligned}$$

where $\{e_1, e_2\}$ is a local orthonormal basis of Σ , $\text{div}_\Sigma \mathbf{X} = \langle \bar{\nabla}_{e_i} \mathbf{X}, e_i \rangle$ and K is the curvature tensor on M .

Proof: As in Section 2, let $\{x_i\}$ be the local normal coordinates around a fixed point p on Σ . The induced metric on $\phi_{t,\epsilon}(\Sigma)$ is

$$g_{ij}(t, \epsilon) = \left\langle \frac{\partial \phi_{t,\epsilon}}{\partial x_i}, \frac{\partial \phi_{t,\epsilon}}{\partial x_j} \right\rangle.$$

For simplicity, we denote $\frac{\partial \phi_{0,0}}{\partial x_i}$ by e_i , $g_{ij}(t, \epsilon)$ by g_{ij} and $\phi_{t,\epsilon}$ by ϕ . It is easy to see that

$$(4.2) \quad \frac{\partial}{\partial t} \big|_{t=0, \epsilon=0} g_{ij} = \langle \bar{\nabla}_{e_i} \mathbf{X}, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} \mathbf{X} \rangle,$$

$$(4.3) \quad \frac{\partial}{\partial \epsilon} \big|_{t=0, \epsilon=0} g_{ij} = \langle \bar{\nabla}_{e_i} \mathbf{Y}, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} \mathbf{Y} \rangle,$$

and

$$(4.4) \quad \begin{aligned} \frac{\partial^2}{\partial t \partial \epsilon} \big|_{t=0, \epsilon=0} g_{ij} &= \langle \bar{\nabla}_{e_i} \mathbf{Z} + K(\mathbf{Y}, e_i) \mathbf{X}, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} \mathbf{Z} + K(\mathbf{Y}, e_j) \mathbf{X} \rangle \\ &\quad + \langle \bar{\nabla}_{e_i} \mathbf{X}, \bar{\nabla}_{e_j} \mathbf{Y} \rangle + \langle \bar{\nabla}_{e_i} \mathbf{Y}, \bar{\nabla}_{e_j} \mathbf{X} \rangle. \end{aligned}$$

Here, K is the curvature tensor on M . As in Section 2, if we denote

$$\nu_\beta := \nu_{\beta, t, \epsilon} = \frac{\det^{(\beta+1)/2}(g)}{\omega^\beta(\partial\phi/\partial x_1, \partial\phi/\partial x_2)},$$

then

$$L_\beta(\phi_{t,\epsilon}) = \int_\Sigma \nu_\beta dx_1 \wedge dx_2.$$

We have

$$\begin{aligned} \frac{\partial}{\partial t} \nu_\beta &= \frac{\beta+1}{2} \frac{\det^{(\beta+1)/2}(g) g^{ij} \frac{\partial}{\partial t} g_{ij}}{\omega^\beta(\partial\phi/\partial x_1, \partial\phi/\partial x_2)} \\ &\quad - \beta \frac{\det^{(\beta+1)/2}(g) [\omega(\bar{\nabla}_{\partial\phi/\partial x_1} \partial\phi/\partial t, \partial\phi/\partial x_2) + \omega(\partial\phi/\partial x_1, \bar{\nabla}_{\partial\phi/\partial x_2} \partial\phi/\partial t)]}{\omega^{\beta+1}(\partial\phi/\partial x_1, \partial\phi/\partial x_2)} \end{aligned}$$

and

$$\begin{aligned} &\frac{\partial^2}{\partial \epsilon \partial t} \big|_{t=0, \epsilon=0} \nu_\beta \\ &= \frac{(\beta+1)^2}{4} \frac{g^{ij} \frac{\partial}{\partial t} \big|_{t=0, \epsilon=0} g_{ij} g^{kl} \frac{\partial}{\partial \epsilon} \big|_{t=0, \epsilon=0} g_{kl}}{\cos^\beta \alpha} + \frac{\beta+1}{2} \frac{g^{ij} \frac{\partial^2}{\partial t \partial \epsilon} \big|_{t=0, \epsilon=0} g_{ij}}{\cos^\beta \alpha} \\ &\quad - \frac{\beta+1}{2} \frac{\frac{\partial}{\partial t} \big|_{t=0, \epsilon=0} g_{ij} \frac{\partial}{\partial \epsilon} \big|_{t=0, \epsilon=0} g_{ij}}{\cos^\beta \alpha} \\ &\quad - \frac{\beta(\beta+1)}{2} \frac{g^{ij} \frac{\partial}{\partial t} \big|_{t=0, \epsilon=0} g_{ij} [\omega(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial \epsilon} \big|_{t=0, \epsilon=0}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial \epsilon} \big|_{t=0, \epsilon=0})]}{\cos^{\beta+1} \alpha} \\ &\quad - \frac{\beta(\beta+1)}{2} \frac{g^{ij} \frac{\partial}{\partial \epsilon} \big|_{t=0, \epsilon=0} g_{ij} [\omega(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial t} \big|_{t=0, \epsilon=0}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial t} \big|_{t=0, \epsilon=0})]}{\cos^{\beta+1} \alpha} \\ &\quad - \beta \frac{\omega(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial t} \big|_{t=0, \epsilon=0}, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial \epsilon} \big|_{t=0, \epsilon=0}) + \omega(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial \epsilon} \big|_{t=0, \epsilon=0}, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial t} \big|_{t=0, \epsilon=0})}{\cos^{\beta+1} \alpha} \\ &\quad - \beta \frac{\omega(\bar{\nabla}_{e_1} \frac{\partial^2}{\partial t \partial \epsilon} \big|_{t=0, \epsilon=0} + K(\frac{\partial \phi}{\partial \epsilon} \big|_{t=0, \epsilon=0}, e_1) \frac{\partial \phi}{\partial t} \big|_{t=0, \epsilon=0}, e_2)}{\cos^{\beta+1} \alpha} \\ &\quad - \beta \frac{\omega(e_1, \bar{\nabla}_{e_2} \frac{\partial^2}{\partial t \partial \epsilon} \big|_{t=0, \epsilon=0} + K(\frac{\partial \phi}{\partial \epsilon} \big|_{t=0, \epsilon=0}, e_2) \frac{\partial \phi}{\partial t} \big|_{t=0, \epsilon=0})}{\cos^{\beta+1} \alpha} \end{aligned}$$

$$\begin{aligned}
& + \frac{\beta(\beta+1)}{\cos^{\beta+2} \alpha} [\omega(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial t} |_{t=0, \epsilon=0}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial t} |_{t=0, \epsilon=0})] \\
& \cdot [\omega(\bar{\nabla}_{e_1} \frac{\partial \phi}{\partial \epsilon} |_{t=0, \epsilon=0}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \frac{\partial \phi}{\partial \epsilon} |_{t=0, \epsilon=0})].
\end{aligned}$$

By (4.2), (4.3) and (4.4), we have

$$\begin{aligned}
& \frac{\partial^2}{\partial \epsilon \partial t} |_{t=0, \epsilon=0} \nu_\beta \\
= & (\beta+1)^2 \frac{\langle \bar{\nabla}_{e_i} \mathbf{X}, e_i \rangle \langle \bar{\nabla}_{e_j} \mathbf{Y}, e_j \rangle}{\cos^\beta \alpha} + (\beta+1) \frac{\langle \bar{\nabla}_{e_i} \mathbf{Z}, e_i \rangle}{\cos^\beta \alpha} \\
& + (\beta+1) \frac{\langle \bar{\nabla}_{e_i} \mathbf{X}, \bar{\nabla}_{e_i} \mathbf{Y} \rangle}{\cos^\beta \alpha} - (\beta+1) \frac{K(\mathbf{X}, e_i, \mathbf{Y}, e_i)}{\cos^\beta \alpha} \\
& - \frac{\beta+1}{2} \frac{(\langle \bar{\nabla}_{e_i} \mathbf{X}, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} \mathbf{X} \rangle)(\langle \bar{\nabla}_{e_i} \mathbf{Y}, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} \mathbf{Y} \rangle)}{\cos^\beta \alpha} \\
& - \beta(\beta+1) \frac{\langle \bar{\nabla}_{e_i} \mathbf{X}, e_i \rangle [\omega(\bar{\nabla}_{e_1} \mathbf{Y}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{Y})]}{\cos^{\beta+1} \alpha} \\
& - \beta(\beta+1) \frac{\langle \bar{\nabla}_{e_i} \mathbf{Y}, e_i \rangle [\omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X})]}{\cos^{\beta+1} \alpha} \\
& - \beta \frac{\omega(\bar{\nabla}_{e_1} \mathbf{X}, \bar{\nabla}_{e_2} \mathbf{Y}) + \omega(\bar{\nabla}_{e_1} \mathbf{Y}, \bar{\nabla}_{e_2} \mathbf{X})}{\cos^{\beta+1} \alpha} \\
& - \beta \frac{\omega(\bar{\nabla}_{e_1} \mathbf{Z}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{Z})}{\cos^{\beta+1} \alpha} \\
& - \beta \frac{\omega(K(\mathbf{Y}, e_1) \mathbf{X}, e_2) + \omega(e_1, K(\mathbf{Y}, e_2) \mathbf{X})}{\cos^{\beta+1} \alpha} \\
(4.5) \quad & + \beta(\beta+1) \frac{[\omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X})][\omega(\bar{\nabla}_{e_1} \mathbf{Y}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{Y})]}{\cos^{\beta+2} \alpha}.
\end{aligned}$$

Since Σ is a β -symplectic critical surface, by the first variation formula (2.4), we see that

$$(4.6) \quad (\beta+1) \int_{\Sigma} \frac{\langle \bar{\nabla}_{e_i} \mathbf{Z}, e_i \rangle}{\cos^\beta \alpha} d\mu - \beta \int_{\Sigma} \frac{\omega(\bar{\nabla}_{e_1} \mathbf{Z}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{Z})}{\cos^{\beta+1} \alpha} d\mu = 0.$$

Notice that,

$$\begin{aligned}
& 2\langle \bar{\nabla}_{e_i} \mathbf{X}, \bar{\nabla}_{e_i} \mathbf{Y} \rangle - (\langle \bar{\nabla}_{e_i} \mathbf{X}, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} \mathbf{X} \rangle)(\langle \bar{\nabla}_{e_i} \mathbf{Y}, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} \mathbf{Y} \rangle) \\
(4.7) \quad & = 2\langle \bar{\nabla}_{e_i}^\perp \mathbf{X}, \bar{\nabla}_{e_i}^\perp \mathbf{Y} \rangle - (\langle e_i, \bar{\nabla}_{e_j} \mathbf{X} \rangle \langle e_j, \bar{\nabla}_{e_i} \mathbf{Y} \rangle + \langle e_i, \bar{\nabla}_{e_j} \mathbf{Y} \rangle \langle e_j, \bar{\nabla}_{e_i} \mathbf{X} \rangle),
\end{aligned}$$

and

$$(4.8) \quad \langle \bar{\nabla}_{e_i} \mathbf{X}, e_i \rangle = \operatorname{div}_{\Sigma} \mathbf{X}.$$

Putting (4.6), (4.7) and (4.8) into (4.5), we obtain the identity in the lemma. Q.E.D.

Assume now that $\mathbf{X} = \mathbf{Y}$ is a normal vector field, then we have

$$(4.9) \quad \operatorname{div}_{\Sigma} \mathbf{X} = -\langle \mathbf{X}, \mathbf{H} \rangle,$$

$$(4.10) \quad \langle e_i, \bar{\nabla}_{e_j} \mathbf{X} \rangle \langle e_j, \bar{\nabla}_{e_i} \mathbf{X} \rangle = \langle \mathbf{X}, \mathbf{A}(e_i, e_j) \rangle^2,$$

and

$$\begin{aligned}
& -\beta \int_{\Sigma} \frac{\omega(K(\mathbf{X}, e_1) \mathbf{X}, e_2) + \omega(e_1, K(\mathbf{X}, e_2) \mathbf{X})}{\cos^{\beta+1} \alpha} d\mu - 2\beta \int_{\Sigma} \frac{\omega(\bar{\nabla}_{e_1} \mathbf{X}, \bar{\nabla}_{e_2} \mathbf{X})}{\cos^{\beta+1} \alpha} d\mu \\
& = \beta \int_{\Sigma} \frac{\langle J e_2, K(\mathbf{X}, e_1) \mathbf{X} \rangle - \langle J e_1, K(\mathbf{X}, e_2) \mathbf{X} \rangle}{\cos^{\beta+1} \alpha} d\mu \\
& \quad -\beta \int_{\Sigma} \frac{e_1(\omega(\mathbf{X}, \bar{\nabla}_{e_2} \mathbf{X})) - \omega(\mathbf{X}, \bar{\nabla}_{e_1} \bar{\nabla}_{e_2} \mathbf{X})}{\cos^{\beta+1} \alpha} d\mu \\
& \quad -\beta \int_{\Sigma} \frac{e_2(\omega(\bar{\nabla}_{e_1} \mathbf{X}, \mathbf{X})) - \omega(\bar{\nabla}_{e_2} \bar{\nabla}_{e_1} \mathbf{X}, \mathbf{X})}{\cos^{\beta+1} \alpha} d\mu \\
& = \beta \int_{\Sigma} \frac{K(\mathbf{X}, e_1, J e_2, \mathbf{X}) + K(\mathbf{X}, e_2, \mathbf{X}, J e_1) + K(e_1, e_2, J \mathbf{X}, \mathbf{X})}{\cos^{\beta+1} \alpha} d\mu \\
(4.11) \quad & -\beta(\beta+1) \int_{\Sigma} \frac{\omega(\mathbf{X}, \bar{\nabla}_{e_2} \mathbf{X}) \nabla_{e_1} \cos \alpha + \omega(\bar{\nabla}_{e_1} \mathbf{X}, \mathbf{X}) \nabla_{e_2} \cos \alpha}{\cos^{\beta+2} \alpha} d\mu.
\end{aligned}$$

In the last step, we used integration by parts. By the first Bianchi identity, we have

$$\begin{aligned}
& K(\mathbf{X}, e_1, J e_2, \mathbf{X}) + K(\mathbf{X}, e_2, \mathbf{X}, J e_1) + K(e_1, e_2, J \mathbf{X}, \mathbf{X}) \\
(4.12) \quad & = K(\mathbf{X}, e_1, J \mathbf{X}, e_2) + K(\mathbf{X}, e_2, e_1, J \mathbf{X}) + K(\mathbf{X}, J \mathbf{X}, e_2, e_1) = 0.
\end{aligned}$$

Therefore, by (4.9), (4.10), (4.11) and (4.12), we obtain:

Corollary 4.2. *If \mathbf{X} is a normal variation, then*

$$\begin{aligned}
& \frac{\partial^2}{\partial t^2} \big|_{t=0} L_{\beta}(\phi_t) := II_{\beta}(\mathbf{X}) \\
& = (\beta+1) \int_{\Sigma} \frac{|\bar{\nabla}_{e_i}^{\perp} \mathbf{X}|^2}{\cos^{\beta} \alpha} d\mu - (\beta+1) \int_{\Sigma} \frac{\langle \mathbf{X}, \mathbf{A}(e_i, e_j) \rangle^2}{\cos^{\beta} \alpha} d\mu - (\beta+1) \int_{\Sigma} \frac{K(\mathbf{X}, e_i, \mathbf{X}, e_i)}{\cos^{\beta} \alpha} d\mu \\
& \quad + (\beta+1)^2 \int_{\Sigma} \frac{\langle \mathbf{X}, \mathbf{H} \rangle^2}{\cos^{\beta} \alpha} d\mu + 2\beta(\beta+1) \int_{\Sigma} \frac{\langle \mathbf{X}, \mathbf{H} \rangle (\omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X}))}{\cos^{\beta+1} \alpha} d\mu \\
& \quad -\beta(\beta+1) \int_{\Sigma} \frac{\omega(\mathbf{X}, \bar{\nabla}_{e_2} \mathbf{X}) \nabla_{e_1} \cos \alpha + \omega(\bar{\nabla}_{e_1} \mathbf{X}, \mathbf{X}) \nabla_{e_2} \cos \alpha}{\cos^{\beta+2} \alpha} d\mu \\
(4.13) \quad & + (\beta+1) \int_{\Sigma} \frac{(\omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X}))^2}{\cos^{\beta+2} \alpha} d\mu.
\end{aligned}$$

Next, we will follow Micallef-Wolfson's idea ([5]) to give another version of the second variation formula so that we can have some rigidity results as its corollaries. For the background material of the geometry of a surface in a four-manifold, we refer to Section 3 of Micallef-Wolfson's paper ([5]).

Set $\mathbf{X} = x_3 e_3 + x_4 e_4$, then

$$\omega(\mathbf{X}, \bar{\nabla}_{e_2} \mathbf{X}) = \langle J \mathbf{X}, \bar{\nabla}_{e_2} \mathbf{X} \rangle$$

$$\begin{aligned}
&= \langle x_3 J e_3 + x_4 J e_4, x_{32} e_3 + x_{42} e_4 - x_3 h_{2i}^3 e_i - x_4 h_{2l}^4 e_l \rangle \\
&= x_3 x_{42} \langle J e_3, e_4 \rangle + x_4 x_{32} \langle J e_4, e_3 \rangle \\
&\quad + x_3 (x_3 h_{12}^3 + x_4 h_{12}^4) \langle e_3, J e_1 \rangle + x_3 (x_3 h_{22}^3 + x_4 h_{22}^4) \langle e_3, J e_2 \rangle \\
&\quad + x_4 (x_3 h_{12}^3 + x_4 h_{12}^4) \langle e_4, J e_1 \rangle + x_4 (x_3 h_{22}^3 + x_4 h_{22}^4) \langle e_4, J e_2 \rangle.
\end{aligned}$$

Notice that (see (2.6))

$$(4.14) \quad \langle J e_1, e_2 \rangle = \langle J e_3, e_4 \rangle = \cos \alpha,$$

$$(4.15) \quad \langle J e_1, e_3 \rangle = -\langle J e_2, e_4 \rangle,$$

$$(4.16) \quad \langle J e_1, e_4 \rangle = \langle J e_2, e_3 \rangle.$$

Thus, we have

$$\begin{aligned}
\omega(\mathbf{X}, \bar{\nabla}_{e_2} \mathbf{X}) &= (x_3 x_{42} - x_4 x_{32}) \cos \alpha \\
&\quad + (x_3^2 h_{12}^3 + x_3 x_4 h_{12}^4 - x_3 x_4 h_{22}^3 - x_4^2 h_{22}^4) \langle e_3, J e_1 \rangle \\
(4.17) \quad &\quad + (x_3^2 h_{22}^3 + x_3 x_4 h_{22}^4 + x_3 x_4 h_{12}^3 + x_4^2 h_{12}^4) \langle e_3, J e_2 \rangle,
\end{aligned}$$

and

$$\begin{aligned}
\omega(\mathbf{X}, \bar{\nabla}_{e_1} \mathbf{X}) &= (x_3 x_{41} - x_4 x_{31}) \cos \alpha \\
&\quad + (x_3^2 h_{11}^3 + x_3 x_4 h_{11}^4 - x_3 x_4 h_{12}^3 - x_4^2 h_{12}^4) \langle e_3, J e_1 \rangle \\
(4.18) \quad &\quad + (x_3^2 h_{12}^3 + x_3 x_4 h_{12}^4 + x_3 x_4 h_{11}^3 + x_4^2 h_{11}^4) \langle e_3, J e_2 \rangle,
\end{aligned}$$

Similarly,

$$\begin{aligned}
(4.19) \quad \omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X}) &= -\cos \alpha (x_3 H^3 + x_4 H^4) \\
&\quad + (x_{32} + x_{41}) \langle J e_1, e_3 \rangle + (x_{42} - x_{31}) \langle J e_2, e_3 \rangle.
\end{aligned}$$

By (4.19), we get that

$$\begin{aligned}
&2\beta(\beta+1) \int_{\Sigma} \frac{\langle \mathbf{X}, \mathbf{H} \rangle (\omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X}))}{\cos^{\beta+1} \alpha} d\mu \\
&+ \beta(\beta+1) \int_{\Sigma} \frac{(\omega(\bar{\nabla}_{e_1} \mathbf{X}, e_2) + \omega(e_1, \bar{\nabla}_{e_2} \mathbf{X}))^2}{\cos^{\beta+2} \alpha} d\mu \\
&= -\beta(\beta+1) \int_{\Sigma} \frac{(x_3 H^3 + x_4 H^4)^2}{\cos^{\beta} \alpha} d\mu + \beta(\beta+1) \int_{\Sigma} \frac{(x_{32} + x_{41})^2 \langle J e_1, e_3 \rangle^2}{\cos^{\beta+2} \alpha} d\mu \\
&\quad + \beta(\beta+1) \int_{\Sigma} \frac{(x_{42} - x_{31})^2 \langle J e_2, e_3 \rangle^2}{\cos^{\beta+2} \alpha} d\mu \\
(4.20) \quad &+ 2\beta(\beta+1) \int_{\Sigma} \frac{(x_{32} + x_{41})(x_{42} - x_{31}) \langle J e_1, e_3 \rangle \langle J e_2, e_3 \rangle}{\cos^{\beta+2} \alpha} d\mu.
\end{aligned}$$

Now $II_{\beta}(\mathbf{X})$ can be written as

$$II_{\beta}(\mathbf{X}) = (\beta+1) \int_{\Sigma} \frac{x_{3i}^2 + x_{4i}^2}{\cos^{\beta} \alpha} d\mu - (\beta+1) \int_{\Sigma} \frac{x_3^2 K_{3i3i} + x_4^2 K_{4i4i} + 2x_3 x_4 K_{3i4i}}{\cos^{\beta} \alpha} d\mu$$

$$\begin{aligned}
& -2(\beta+1) \int_{\Sigma} \frac{(x_3 h_{12}^3 + x_4 h_{12}^4)^2 - (x_3 h_{11}^3 + x_4 h_{11}^4)(x_3 h_{22}^3 + x_4 h_{22}^4)}{\cos^{\beta} \alpha} d\mu \\
& + \beta(\beta+1) \int_{\Sigma} \frac{(x_{32} + x_{41})^2 \langle J e_1, e_3 \rangle^2 + (x_{42} - x_{31})^2 \langle J e_2, e_3 \rangle^2}{\cos^{\beta+2} \alpha} d\mu \\
& + 2\beta(\beta+1) \int_{\Sigma} \frac{(x_{32} + x_{41})(x_{42} - x_{31}) \langle J e_1, e_3 \rangle \langle J e_2, e_3 \rangle}{\cos^{\beta+2} \alpha} d\mu \\
(4.21) \quad & + III_{\beta}(\mathbf{X}),
\end{aligned}$$

where

$$III_{\beta}(\mathbf{X}) = -\beta(\beta+1) \int_{\Sigma} \frac{\omega(\mathbf{X}, \bar{\nabla}_{e_2} \mathbf{X}) \nabla_{e_1} \cos \alpha + \omega(\bar{\nabla}_{e_1} \mathbf{X}, \mathbf{X}) \nabla_{e_2} \cos \alpha}{\cos^{\beta+2} \alpha} d\mu$$

In addition, if we set $\mathbf{Y} = -J_{\nu} \mathbf{X} = x_4 e_3 - x_3 e_4$, then by (4.21), we get that

$$\begin{aligned}
& II_{\beta}(\mathbf{X}) + II_{\beta}(\mathbf{Y}) \\
= & 2(\beta+1) \int_{\Sigma} \frac{x_{3i}^2 + x_{4i}^2}{\cos^{\beta} \alpha} d\mu - (\beta+1) \int_{\Sigma} \frac{|\mathbf{X}|^2 (K_{3i3i} + K_{4i4i})}{\cos^{\beta} \alpha} d\mu \\
& - 2(\beta+1) \int_{\Sigma} \frac{|\mathbf{X}|^2 [(h_{12}^3)^2 + (h_{12}^4)^2 - h_{11}^3 h_{22}^3 - h_{11}^4 h_{22}^4]}{\cos^{\beta} \alpha} d\mu \\
& + \beta(\beta+1) \int_{\Sigma} \frac{[(x_{32} + x_{41})^2 + (x_{42} - x_{31})^2] [\langle J e_1, e_3 \rangle^2 + \langle J e_2, e_3 \rangle^2]}{\cos^{\beta+2} \alpha} d\mu \\
(4.22) \quad & + III_{\beta}(\mathbf{X}) + III_{\beta}(\mathbf{Y}).
\end{aligned}$$

Using (4.17) and (4.18), it is easy to check that

$$\begin{aligned}
(4.23) \quad III_{\beta}(\mathbf{X}) + III_{\beta}(\mathbf{Y}) = & 2(\beta+1) \int_{\Sigma} (x_3 x_{42} - x_4 x_{32}) \nabla_{e_1} \frac{1}{\cos^{\beta} \alpha} d\mu \\
& - 2(\beta+1) \int_{\Sigma} (x_3 x_{41} - x_4 x_{31}) \nabla_{e_2} \frac{1}{\cos^{\beta} \alpha} d\mu \\
& - \beta(\beta+1) \int_{\Sigma} \frac{|\mathbf{X}|^2 (h_{12}^3 - h_{22}^4) \langle e_3, J e_1 \rangle \nabla_{e_1} \cos \alpha}{\cos^{\beta+2} \alpha} d\mu \\
& - \beta(\beta+1) \int_{\Sigma} \frac{|\mathbf{X}|^2 (h_{22}^3 + h_{12}^4) \langle e_4, J e_1 \rangle \nabla_{e_1} \cos \alpha}{\cos^{\beta+2} \alpha} d\mu \\
& + \beta(\beta+1) \int_{\Sigma} \frac{|\mathbf{X}|^2 (h_{11}^3 - h_{12}^4) \langle e_3, J e_1 \rangle \nabla_{e_2} \cos \alpha}{\cos^{\beta+2} \alpha} d\mu \\
& + \beta(\beta+1) \int_{\Sigma} \frac{|\mathbf{X}|^2 (h_{12}^3 + h_{11}^4) \langle e_4, J e_1 \rangle \nabla_{e_2} \cos \alpha}{\cos^{\beta+2} \alpha} d\mu.
\end{aligned}$$

Integrating the first two terms in (4.23) by parts, we get that

$$\begin{aligned}
& 2(\beta+1) \int_{\Sigma} (x_3 x_{42} - x_4 x_{32}) \nabla_{e_1} \frac{1}{\cos^{\beta} \alpha} d\mu - 2(\beta+1) \int_{\Sigma} (x_3 x_{41} - x_4 x_{31}) \nabla_{e_2} \frac{1}{\cos^{\beta} \alpha} d\mu \\
= & 2(\beta+1) \int_{\Sigma} \frac{2(x_{32} x_{41} - x_{31} x_{42}) + x_3 (x_{412} - x_{421}) + x_4 (x_{321} - x_{312})}{\cos^{\beta} \alpha} d\mu
\end{aligned}$$

$$= 4(\beta + 1) \int_{\Sigma} \frac{x_{32}x_{41} - x_{31}x_{42}}{\cos^{\beta} \alpha} d\mu + 2(\beta + 1) \int_{\Sigma} \frac{|\mathbf{X}|^2 R_{3412}}{\cos^{\beta} \alpha} d\mu,$$

where R_{3412} is the normal curvature of Σ in M . Here, we have used the fact that (see (3.9) of ([5]))

$$-x_3 R_{3412} = x_{421} - x_{412}, \quad x_4 R_{3412} = x_{321} - x_{312}.$$

We will compute the last four terms of (4.23) at each point, so we can choose an orthonormal frame $\{e_1, e_2, e_3, e_4\}$ around the point so that J takes the form

$$(4.24) \quad J = \begin{pmatrix} 0 & \cos \alpha & \sin \alpha & 0 \\ -\cos \alpha & 0 & 0 & -\sin \alpha \\ -\sin \alpha & 0 & 0 & \cos \alpha \\ 0 & \sin \alpha & -\cos \alpha & 0 \end{pmatrix}.$$

By the equation of a β -symplectic critical surface (2.11), we have

$$H^4 = \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_1} \alpha, \quad H^3 = \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_2} \alpha.$$

Using (2.12), we have

$$\begin{aligned} & -(h_{12}^3 - h_{22}^4) \nabla_{e_1} \cos \alpha + (h_{11}^3 - h_{12}^4) \nabla_{e_2} \cos \alpha \\ = & \sin \alpha (h_{12}^3 + h_{11}^4 - H^4) \nabla_{e_1} \alpha + \sin \alpha (h_{22}^3 + h_{12}^4 - H^3) \nabla_{e_2} \alpha \\ = & -\sin \alpha (\nabla_{e_1} \alpha + \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_1} \alpha) \nabla_{e_1} \alpha - \sin \alpha (\nabla_{e_2} \alpha + \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_2} \alpha) \nabla_{e_2} \alpha \\ = & -\frac{\sin \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^2 \alpha} |\nabla \alpha|^2, \end{aligned}$$

and

$$-(h_{22}^3 + h_{12}^4) \nabla_{e_1} \cos \alpha + (h_{12}^3 + h_{11}^4) \nabla_{e_2} \cos \alpha = -\sin \alpha \nabla_{e_2} \alpha \nabla_{e_1} \alpha + \sin \alpha \nabla_{e_1} \alpha \nabla_{e_2} \alpha = 0.$$

By the Ricci equation,

$$R_{3412} = K_{3412} + h_{1k}^3 h_{2k}^4 - h_{2k}^3 h_{1k}^4,$$

we obtain that

$$\begin{aligned} III_{\beta}(\mathbf{X}) + III_{\beta}(\mathbf{Y}) &= 4(\beta + 1) \int_{\Sigma} \frac{x_{32}x_{41} - x_{31}x_{42}}{\cos^{\beta} \alpha} d\mu + 2(\beta + 1) \int_{\Sigma} \frac{|\mathbf{X}|^2 K_{3412}}{\cos^{\beta} \alpha} d\mu \\ &\quad + 2(\beta + 1) \int_{\Sigma} \frac{|\mathbf{X}|^2 (h_{1k}^3 h_{2k}^4 - h_{2k}^3 h_{1k}^4)}{\cos^{\beta} \alpha} d\mu \\ (4.25) \quad &\quad - \beta(\beta + 1) \int_{\Sigma} \frac{|\mathbf{X}|^2 \sin^2 \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^{\beta+4} \alpha} |\nabla \alpha|^2 d\mu. \end{aligned}$$

Putting (4.25) into (4.22) yields

$$\begin{aligned} & II_{\beta}(\mathbf{X}) + II_{\beta}(\mathbf{Y}) \\ = & 2(\beta + 1) \int_{\Sigma} \frac{(x_{32} + x_{41})^2 + (x_{42} - x_{31})^2}{\cos^{\beta} \alpha} d\mu \end{aligned}$$

$$\begin{aligned}
& -(\beta+1) \int_{\Sigma} \frac{|\mathbf{X}|^2 (K_{3i3i} + K_{4i4i} - 2K_{1234})}{\cos^{\beta} \alpha} d\mu \\
& +\beta(\beta+1) \int_{\Sigma} \frac{[(x_{32} + x_{41})^2 + (x_{42} - x_{31})^2] \sin^2 \alpha}{\cos^{\beta+2} \alpha} d\mu \\
& -\beta(\beta+1) \int_{\Sigma} \frac{|\mathbf{X}|^2 \sin^2 \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^{\beta+4} \alpha} |\nabla \alpha|^2 d\mu \\
& -2(\beta+1) \int_{\Sigma} \frac{|\mathbf{X}|^2 [(h_{12}^3)^2 + (h_{12}^4)^2 - h_{11}^3 h_{22}^3 - h_{11}^4 h_{22}^4 - h_{1k}^3 h_{2k}^4 + h_{2k}^3 h_{1k}^4]}{\cos^{\beta} \alpha} d\mu.
\end{aligned}$$

By Lemma 3.2 of [4], it is easy to check that

$$K_{3i3i} + K_{4i4i} - 2K_{1234} = 2K \sin^2 \alpha,$$

where R is the scalar curvature of M . By (3.16) of [5], we have

$$(x_{32} + x_{41})^2 + (x_{42} - x_{31})^2 = |\bar{\partial} \mathbf{X}|^2.$$

Using the fact that Σ is a β -symplectic critical surface again, we can see that

$$\begin{aligned}
& (h_{12}^3)^2 + (h_{12}^4)^2 - h_{11}^3 h_{22}^3 - h_{11}^4 h_{22}^4 - h_{1k}^3 h_{2k}^4 + h_{2k}^3 h_{1k}^4 \\
& = (h_{11}^4 + h_{12}^3)(h_{12}^3 - h_{22}^4) + (h_{12}^4 + h_{22}^3)(h_{12}^4 - h_{11}^3) \\
& = -\nabla_{e_1} \alpha (-\nabla_{e_1} \alpha - H^4) - \nabla_{e_2} \alpha (-\nabla_{e_2} \alpha - H^3) \\
& = \nabla_{e_1} \alpha (\nabla_{e_1} \alpha + \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_1} \alpha) + \nabla_{e_2} \alpha (\nabla_{e_2} \alpha + \beta \frac{\sin^2 \alpha}{\cos^2 \alpha} \nabla_{e_2} \alpha) \\
& = \frac{\cos^2 \alpha + \beta \sin^2 \alpha}{\cos^2 \alpha} |\nabla \alpha|^2
\end{aligned}$$

Therefore, we can conclude that

$$\begin{aligned}
II_{\beta}(\mathbf{X}) + II_{\beta}(\mathbf{Y}) & = -2(\beta+1) \int_{\Sigma} \frac{K|\mathbf{X}|^2 \sin^2 \alpha}{\cos^{\beta} \alpha} d\mu + 2(\beta+1) \int_{\Sigma} \frac{|\bar{\partial} \mathbf{X}|^2}{\cos^{\beta} \alpha} d\mu \\
& +\beta(\beta+1) \int_{\Sigma} \frac{|\bar{\partial} \mathbf{X}|^2 \sin^2 \alpha}{\cos^{\beta+2} \alpha} d\mu \\
& -\beta(\beta+1) \int_{\Sigma} \frac{|\mathbf{X}|^2 \sin^2 \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^{\beta+4} \alpha} |\nabla \alpha|^2 d\mu \\
& -2(\beta+1) \int_{\Sigma} \frac{|\mathbf{X}|^2 (\cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^{\beta+2} \alpha} |\nabla \alpha|^2 d\mu \\
& = -2(\beta+1) \int_{\Sigma} \frac{K|\mathbf{X}|^2 \sin^2 \alpha}{\cos^{\beta} \alpha} d\mu + (\beta+1) \int_{\Sigma} \frac{|\bar{\partial} \mathbf{X}|^2 (2 \cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^{\beta+2} \alpha} d\mu \\
& -(\beta+1) \int_{\Sigma} \frac{(2 \cos^2 \alpha + \beta \sin^2 \alpha) (\cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^{\beta+4} \alpha} |\mathbf{X}|^2 |\nabla \alpha|^2 d\mu.
\end{aligned}$$

This gives the following theorem:

Theorem 4.3. *If we choose $\mathbf{X} = x_3e_3 + x_4e_4$ and $\mathbf{Y} = -J_\nu\mathbf{X} = x_4e_3 - x_3e_4$, then the second variation of the functional L_β of a β -symplectic critical surface is*

$$\begin{aligned}
 & II_\beta(\mathbf{X}) + II_\beta(\mathbf{Y}) \\
 &= -2(\beta + 1) \int_\Sigma \frac{K|\mathbf{X}|^2 \sin^2 \alpha}{\cos^\beta \alpha} d\mu + (\beta + 1) \int_\Sigma \frac{|\bar{\partial}\mathbf{X}|^2 (2 \cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^{\beta+2} \alpha} d\mu \\
 (4.26) \quad & -(\beta + 1) \int_\Sigma \frac{(2 \cos^2 \alpha + \beta \sin^2 \alpha)(\cos^2 \alpha + \beta \sin^2 \alpha)}{\cos^{\beta+4} \alpha} |\mathbf{X}|^2 |\nabla \alpha|^2 d\mu.
 \end{aligned}$$

As applications of the stability inequality (4.26), we can obtain some rigidity results for stable β -symplectic critical surfaces.

Corollary 4.4. *Let M be a Kähler surface with positive scalar curvature K . If Σ is a stable β -symplectic critical surface in M with $\beta \geq 0$, whose normal bundle admits a nontrivial section \mathbf{X} with*

$$\frac{|\bar{\partial}\mathbf{X}|^2}{|\mathbf{X}|^2} \leq \frac{\cos^2 \alpha + \beta \sin^2 \alpha}{\cos^2 \alpha} |\nabla \alpha|^2,$$

then Σ is a holomorphic curve.

Proof: If Σ is a stable β -symplectic critical surface in M , we have

$$II_\beta(\mathbf{X}) + II_\beta(\mathbf{Y}) \geq 0,$$

where $\mathbf{Y} = -J_\nu\mathbf{X}$. By Theorem 4.3 with $\beta \geq 0$, we obtain

$$2(\beta + 1) \int_\Sigma \frac{K|\mathbf{X}|^2 \sin^2 \alpha}{\cos^\beta \alpha} d\mu \leq 0,$$

Since $\beta \geq 0$ and $K > 0$, by the last inequality, we must have $\sin \alpha \equiv 0$, that is, Σ is a holomorphic curve. Q.E.D.

Corollary 4.5. *Let M be a Kähler surface with positive scalar curvature K . If Σ is a stable β -symplectic critical surface in M with $\beta \geq 0$ and $\chi(\nu) \geq g$, where $\chi(\nu)$ is the Euler characteristic of the normal bundle ν of Σ in M and g is the genus of Σ , then Σ is a holomorphic curve.*

Proof: Let L be the canonical line bundle over Σ , let $H^0(\Sigma, \mathcal{O}(\nu))$ denote the space of holomorphic sections of ν , and let $H^0(\Sigma, \mathcal{O}(\nu \otimes L))$ denote the space of holomorphic sections of $\nu \otimes L$, then by Riemann-Roch theorem, we have

$$\dim H^0(\Sigma, \mathcal{O}(\nu)) = \chi(\nu) - g + 1 + \dim H^0(\Sigma, \mathcal{O}(\nu \otimes L)) \geq 1.$$

We therefore have a nontrivial holomorphic section on ν and the corollary follows from Corollary 4.4. Q.E.D.

5. CONTINUITY METHOD AND OPENNESS

We would like to study the family of β -symplectic critical surfaces in $[0, \infty)$. We set

$$S := \{\beta \in [0, \infty) \mid \exists \text{ a strictly stable } \beta - \text{symplectic critical surface}\}.$$

Theorem 5.1. *S is open on $[0, \infty)$. In other words, if there is a strictly stable β_0 -symplectic critical surface Σ_{β_0} , then there is a strictly stable β -symplectic critical surface in the neighborhood of Σ_{β_0} for β sufficiently close to β_0 .*

Before we prove our theorem, we state theorem 1.1 of White [8] here for the reader's convenience.

Theorem 5.2. [8] *Let M be a smooth compact m -dimensional Riemannian manifold, V be a smooth Euclidean vector bundle over M , and ∇ be a smooth Riemannian connection on V . Let G be the Banach space of C^q functions f that assign to each $x \in M$, $v \in V_x$, and linear map $L : \text{Tan}_x M \rightarrow V_x$ a real number $f(x, v, L)$ in such a way that $D_3 f$ is also C^q . Let Γ be an open subset of a Banach space and A be a smooth map from Γ to G . Then:*

(1) *If $u_t \in C^{2,\alpha}(M, V)$ is a differentiable one-parameter family of sections, then*

$$\left(\frac{d}{dt}\right)_{t=0} \int_M A_\gamma(x, u_t(x), \nabla u_t(x)) dx = \int_M H(\gamma, u_0)(x) \cdot \frac{d}{dt} u_0(x) dx$$

where

$$H(\gamma, u) = -\text{div} D_3 A_\gamma(x, u, \nabla u) + D_2 A_\gamma(x, u, \nabla u).$$

Furthermore, if $2 \leq j \leq q$ and $u \in C^{j,\alpha}(M, V)$, then $H(\gamma, u) \in C^{j-2,\alpha}(M, V)$ and the map

$$H : \Gamma \times C^{j,\alpha}(M, V) \rightarrow C^{j-2,\alpha}(M, V)$$

is C^{q-j} .

(2) *The linearization*

$$J = D_2 H(\gamma_0, u_0) : C^{j,\alpha}(M, V) \rightarrow C^{j-2,\alpha}(M, V)$$

is a self-adjoint second order linear partial differential operator.

If in addition J is an elliptic operator, then

(3) *J is a Fredholm map with Fredholm index 0.*

(4) *If $H(\gamma_0, u_0) = 0$, then u_0 must be $C^{q,\beta}$ for every $\beta < 1$.*

(5) *There is an orthonormal basis for $L^2(M, V)$ consisting of C^q eigenfunctions for J . For every λ , only finitely many eigenfunctions have eigenvalues less than λ .*

Proof of Theorem 5.1: It suffices to show that if $\beta_0 \in S$, i.e., there exists a strictly stable β_0 -symplectic critical surface Σ_{β_0} in M , then there at least one solution to the equation (2.2) which is also strictly stable when β is sufficiently close to β_0 . We will use the above theorem and Implicit Function Theorem.

In order to use the above theorem of White. We need equate the map F with the section u of normal bundle and we consider the equivalent class $[u]$ in place of u ([u]

denote the set of all maps $u \circ \phi$ where $\phi : \Sigma \rightarrow \Sigma$ is a diffeomorphism). See theorem 2.1 in [8].

We have seen in Section 3 that the linearization $J_{\beta_0} = DP(\beta, [F])$ is an elliptic operator. Therefore, by (3) of the above theorem, J_{β_0} is a Fredholm operator with Fredholm index 0. Since we assume that Σ_{β_0} is strictly stable so that J_{β_0} has no normal Jacobi fields, therefore J_{β_0} is an isomorphism. Thus, by Implicit Function Theorem, we see that (2.2) has a solution for $\beta \in (\beta_0 - \delta, \beta_0 + \delta)$ for some $\delta > 0$. We denote it by Σ_β .

It remains to show that Σ_β is strictly stable. This follows from (4.13). By (4.13), we see that the operator J_β is continuous in β . In particular, the eigenvalues of J_β are continuous in β . Since Σ_{β_0} is strictly stable, the smallest eigenvalue of J_{β_0} is positive. By continuity, there exists $\delta_1 > 0$ (which may be smaller than δ above), so that the smallest eigenvalue of J_β is positive for $\beta \in (\beta_0 - \delta_1, \beta_0 + \delta_1)$. That means, Σ_β is a strictly stable β -symplectic critical surface in M for $\beta \in (\beta_0 - \delta_1, \beta_0 + \delta_1)$, namely, $(\beta_0 - \delta_1, \beta_0 + \delta_1) \subset S$. This proves the openness of S . Q.E.D.

6. ROTATIONALLY SYMMETRIC β -SYMPLECTIC CRITICAL SURFACES IN \mathbb{C}^2

In this section, we study the rotationally symmetric β -symplectic critical surfaces in \mathbb{C}^2 of the following form:

$$(6.1) \quad F(r, \theta) = (r \cos \theta, r \sin \theta, f(r), g(r)).$$

Then we have

$$\begin{aligned} e_1 = F_r &= (\cos \theta, \sin \theta, f', g'), & e_2 = F_\theta &= (-r \sin \theta, r \cos \theta, 0, 0), \\ e_3 &= (-f' \cos \theta, -f' \sin \theta, 1, 0), & e_4 &= (-g' \cos \theta, -g' \sin \theta, 0, 1). \end{aligned}$$

Further, we have

$$F_{rr} = (0, 0, f'', g''), \quad F_{r\theta} = (-\sin \theta, \cos \theta, 0, 0), \quad F_{\theta\theta} = (-r \cos \theta, -r \sin \theta, 0, 0).$$

Therefore, the induced metric on Σ is given by

$$\begin{aligned} (g_{ij})_{1 \leq i, j \leq 2} &= \begin{pmatrix} 1 + (f')^2 + (g')^2 & 0 \\ 0 & r^2 \end{pmatrix}, \\ \det(g_{ij}) &= r^2(1 + (f')^2 + (g')^2), \\ (g^{ij})_{1 \leq i, j \leq 2} &= \begin{pmatrix} \frac{1}{1 + (f')^2 + (g')^2} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}, \\ (g_{\alpha\beta})_{3 \leq \alpha, \beta \leq 4} &= \begin{pmatrix} 1 + (f')^2 & f'g' \\ f'g' & 1 + (g')^2 \end{pmatrix}, \\ \det(g_{\alpha\beta}) &= 1 + (f')^2 + (g')^2. \\ (g^{\alpha\beta})_{3 \leq \alpha, \beta \leq 4} &= \frac{1}{1 + (f')^2 + (g')^2} \begin{pmatrix} 1 + (g')^2 & -f'g' \\ -f'g' & 1 + (f')^2 \end{pmatrix}, \end{aligned}$$

Denote by $A = 1 + (f')^2 + (g')^2$, then it is easy to see that

$$(6.2) \quad \cos \alpha = \frac{1}{\sqrt{1 + (f')^2 + (g')^2}} = \frac{1}{\sqrt{A}} > 0,$$

so that the surface Σ is always symplectic. Now we compute \mathbf{H} and $(J(J\nabla \cos \alpha)^T)^\perp$.

$$\begin{aligned} \mathbf{H} &= g^{\alpha\beta} \langle g^{ij} \frac{\partial^2 F}{\partial x_i \partial x_j}, v_\alpha \rangle e_\beta \\ &= [g^{33} \langle g^{11} \frac{\partial^2 F}{\partial r^2} + g^{22} \frac{\partial^2 F}{\partial \theta^2}, e_3 \rangle + g^{34} \langle g^{11} \frac{\partial^2 F}{\partial r^2} + g^{22} \frac{\partial^2 F}{\partial \theta^2}, e_4 \rangle] e_3 \\ &\quad + [g^{34} \langle g^{11} \frac{\partial^2 F}{\partial r^2} + g^{22} \frac{\partial^2 F}{\partial \theta^2}, e_3 \rangle + g^{44} \langle g^{11} \frac{\partial^2 F}{\partial r^2} + g^{22} \frac{\partial^2 F}{\partial \theta^2}, e_4 \rangle] e_4 \\ &= [\frac{1 + (g')^2}{A} (\frac{f''}{A} + \frac{f'}{r}) - \frac{f'g'}{A} (\frac{g''}{A} + \frac{g'}{r})] e_3 \\ &\quad + [\frac{1 + (f')^2}{A} (\frac{g''}{A} + \frac{g'}{r}) - \frac{f'g'}{A} (\frac{f''}{A} + \frac{f'}{r})] e_4 \\ &= \frac{1}{rA^2} [r(1 + (g')^2)f'' - rf'g'g'' + Af'] e_3 \\ (6.3) \quad &+ \frac{1}{rA^2} [r(1 + (f')^2)g'' - rf'g'f'' + Ag'] e_4. \end{aligned}$$

It is clear that

$$\begin{aligned} \nabla \cos \alpha &= (g^{11} \frac{\partial \cos \alpha}{\partial r} + g^{12} \frac{\partial \cos \alpha}{\partial \theta}) e_1 + (g^{21} \frac{\partial \cos \alpha}{\partial r} + g^{22} \frac{\partial \cos \alpha}{\partial \theta}) e_2 \\ &= g^{11} \frac{\partial \cos \alpha}{\partial r} e_1 = -\frac{1}{A^{5/2}} (f'f'' + g'g'') e_1. \end{aligned}$$

Moreover,

$$(Je_1)^T = g^{ij} \langle Je_1, e_i \rangle e_j = g^{22} \langle Je_1, e_2 \rangle e_2 = \frac{1}{r} e_2,$$

$$\begin{aligned} (J(Je_1)^T)^\perp &= \frac{1}{r} (Je_2)^\perp = \frac{1}{r} [g^{\alpha\beta} \langle Je_2, e_\alpha \rangle] e_\beta \\ &= \frac{1}{r} [\frac{1 + (g')^2}{A} rf' - \frac{f'g'}{A} rg'] e_3 + \frac{1}{r} [\frac{1 + (f')^2}{A} rg' - \frac{f'g'}{A} rf'] e_4 \\ &= \frac{f'}{A} e_3 + \frac{g'}{A} e_4. \end{aligned}$$

Thus,

$$(6.4) \quad (J(J\nabla \cos \alpha)^T)^\perp = -\frac{(f'f'' + g'g'')f'}{A^{7/2}} e_3 - \frac{(f'f'' + g'g'')g'}{A^{7/2}} e_4.$$

Putting (6.3) and (6.4) into (2.2)

$$\begin{cases} \frac{1}{rA^{7/2}} [r(1 + (g')^2)f'' - rf'g'g'' + Af'] &= -\beta \frac{f'f'' + g'g''}{A^{7/2}} f' \\ \frac{1}{rA^{7/2}} [r(1 + (f')^2)g'' - rf'g'f'' + Ag'] &= -\beta \frac{f'f'' + g'g''}{A^{7/2}} g' \end{cases}.$$

That is equivalent to

$$\begin{cases} r(1 + (g')^2 + \beta(f')^2)f'' + r(\beta - 1)f'g'g'' + (1 + (f')^2 + (g')^2)f' &= 0 \\ r(1 + (f')^2 + \beta(g')^2)g'' + r(\beta - 1)f'g'f'' + (1 + (f')^2 + (g')^2)g' &= 0 \end{cases}.$$

This implies a nice equation

$$\begin{cases} (rf'(1 + (f')^2 + (g')^2)^{\frac{\beta-1}{2}})' &= 0 \\ (rg'(1 + (f')^2 + (g')^2)^{\frac{\beta-1}{2}})' &= 0 \end{cases}.$$

It is equivalent that

$$(6.5) \quad \begin{cases} rf'(1 + (f')^2 + (g')^2)^{\frac{\beta-1}{2}} &= C_1 \\ rg'(1 + (f')^2 + (g')^2)^{\frac{\beta-1}{2}} &= C_2 \end{cases},$$

for any constant C_1, C_2 .

There is a trivial case for the system (6.5). Actually, if $C_1 = C_2 = 0$, then f, g must be constants and a β -symplectic critical surface must be a plane.

Next, we will divide four cases to study the solutions of (6.5). The easiest one is the case that $\beta = 1$, which follows directly from (6.5):

Theorem 6.1. *Let Σ be a 1-symplectic critical surface (i.e., $\beta = 1$) in \mathbb{C}^2 of the form (6.1), then there exists constants C_1, C_2, C_3 and C_4 , such that*

$$(6.6) \quad f = C_1 \ln r + C_3, \quad g = C_2 \ln r + C_4.$$

When $\beta = 0$, i.e., Σ is a minimal surface in \mathbb{C}^2 , we have

Theorem 6.2. *Let Σ be a minimal surface (i.e., $\beta = 0$) in \mathbb{C}^2 of the form (6.1). If $g \equiv \text{constant}$, we suppose $C_1 = 1$ in (6.5). Then*

$$f(r) = \pm \cosh^{-1} r + C_3,$$

and Σ is the catenoid.

If both f and g are not constants, we suppose $C_1 = C_2 = 1$ in (6.5). Then there exist constants C_3 and C_4 , such that

$$(6.7) \quad f = \pm \cosh^{-1} \frac{r}{\sqrt{2}} + C_3, \quad g = \pm \cosh^{-1} \frac{r}{\sqrt{2}} + C_4.$$

Proof: If $g \equiv \text{constant}$, then the first equation of (6.5) for $\beta = 0$ becomes

$$rf'(1 + (f')^2)^{-\frac{1}{2}} = 1.$$

Rearranging it, we obtain that

$$(f')^2 = \frac{1}{r^2 - 1}.$$

Integrating gives the desired conclusion. The second case follows in the same way since we have $f' = g'$ in this case. Q.E.D.

We will analyze the asymptotic behavior of the solutions. Note that, if one of C_1, C_2 is zero, say, $C_2 = 0$, then from the second equality of (6.5), we see that $g' \equiv 0$. In this case, the first equation in (6.5) becomes a nonlinear equation for f' . We can

analyze it in the same way as we will do in the following theorems in which we deal with the general case.

Now we assume $\beta \neq 1, 0$ and $C_1 \neq 0, C_2 \neq 0$ in the following. The constants C_1, C_2 are determined by the initial data of f' and g' . Without loss of generality, we assume $C_1 = C_2 = 1$.

Theorem 6.3. *For any $\beta > 0$ and $\varepsilon > 0$, the equations*

$$\begin{cases} rf'(1 + (f')^2 + (g')^2)^{\frac{\beta-1}{2}} = 1, \\ rg'(1 + (f')^2 + (g')^2)^{\frac{\beta-1}{2}} = 1, \\ f(\varepsilon) = f_0, \\ g(\varepsilon) = g_0, \end{cases}$$

have a unique C^∞ -solution on $[\varepsilon, +\infty)$. Moreover, $f' = g'$. As $r \rightarrow \infty$, we have the asymptotic expansion,

$$f' = \frac{1}{r} - \frac{\beta-1}{r^3} + o(r^{-3}).$$

As $r \rightarrow 0$, we have the asymptotic expansion,

$$f' = 2^{\frac{1-\beta}{2\beta}} r^{-\frac{1}{\beta}} - \frac{\beta-1}{\beta} 2^{-\frac{3\beta+1}{2\beta}} r^{\frac{1}{\beta}} + o(r^{\frac{1}{\beta}}).$$

Proof: First we show that a solution exists on $[\varepsilon, +\infty)$. Comparing the above equations, we get that $\frac{f'}{g'} = 1$, i.e, $g' = f'$. Putting $f' = g'$ into the second equation we get that $rf'(1 + 2(f')^2)^{\frac{\beta-1}{2}} = 1$. In the interval $[\varepsilon, +\infty)$, f' is bounded. Thus there exists a solution in $[\varepsilon, +\infty)$.

Now we turn to the uniqueness. Assume there is a another solution (\tilde{f}, \tilde{g}) which satisfies the above equations. By the same argument, we have $\tilde{f}' = \tilde{g}'$ and

$$(6.8) \quad f'(1 + 2(f')^2)^{\frac{\beta-1}{2}} - \tilde{f}'(1 + 2(\tilde{f}')^2)^{\frac{\beta-1}{2}} = 0.$$

If $f' > \tilde{f}'$ at some point r_0 , then $(1 + 2(f')^2)^{\frac{\beta-1}{2}} \geq (1 + 2(\tilde{f}')^2)^{\frac{\beta-1}{2}} > 0$ at r_0 when $\beta \geq 1$. That is impossible. Therefore $f' \equiv \tilde{f}'$. Then $f = \tilde{f}$ if they satisfy the same initial condition. Similarly, $g = \tilde{g}$. This proves the uniqueness for $\beta \geq 1$.

On the other hand, if $0 < \beta < 1$, then we can rewrite (6.8) as

$$f'(1 + 2(\tilde{f}')^2)^{\frac{1-\beta}{2}} - \tilde{f}'(1 + 2(f')^2)^{\frac{1-\beta}{2}} = 0.$$

Since $f' > 0, \tilde{f}' > 0$ by the equation, we can rearrange it as

$$(f')^{\frac{2}{1-\beta}} - (\tilde{f}')^{\frac{2}{1-\beta}} + 2(f')^2(\tilde{f}')^2[(f')^{\frac{2\beta}{1-\beta}} - (\tilde{f}')^{\frac{2\beta}{1-\beta}}] = 0.$$

If $f' > \tilde{f}'$ at some point r_0 , then $(f')^{\frac{2}{1-\beta}} > (\tilde{f}')^{\frac{2}{1-\beta}}$ and $(f')^{\frac{2\beta}{1-\beta}} > (\tilde{f}')^{\frac{2\beta}{1-\beta}}$ at r_0 when $0 < \beta < 1$. That is impossible. Therefore $f' \equiv \tilde{f}'$. Then $f = \tilde{f}$ if they satisfy the same initial condition. Similarly, $g = \tilde{g}$. This proves the uniqueness for $0 < \beta < 1$.

Next we study the behaviour of f' which satisfies

$$(6.9) \quad rf'(1 + 2(f')^2)^{\frac{\beta-1}{2}} = 1,$$

as $r \rightarrow \infty$ and $r \rightarrow 0$.

As $r \rightarrow \infty$, f' must tend to 0. By the equation (6.9), we see that $\lim_{r \rightarrow \infty} r f' = 1$. We write $f' = \frac{1}{r} + \psi(r)$, then we obtain that $\lim_{r \rightarrow \infty} r \psi(r) = 0$. Putting it into (6.9),

$$(1 + r\psi)^{\frac{2}{\beta-1}} \left(1 + \frac{2}{r^2} + \frac{4\psi}{r} + 2\psi^2\right) = 1.$$

That implies that

$$\frac{2}{r^2\psi} + \frac{2}{\beta-1}r + O\left(\frac{1}{r}\right) = 0.$$

Therefore,

$$r^3\psi = -(\beta-1) + o(1),$$

which implies the desired expansion.

As $r \rightarrow 0$, f' must tend to infinity. By the equation (6.9), we can easily see that $\lim_{r \rightarrow 0} 2^{\frac{\beta-1}{2}} r (f')^\beta = 1$, which implies that $\lim_{r \rightarrow 0} r^{\frac{1}{\beta}} f' = 2^{\frac{1-\beta}{2\beta}}$. Now we can write $r^{\frac{1}{\beta}} f' = 2^{\frac{1-\beta}{2\beta}} + w(r)$, where $\lim_{r \rightarrow 0} w(r) = 0$. Putting it into (6.9), we obtain that

$$(1 + 2^{\frac{\beta-1}{2\beta}} w(r))^{\frac{2}{\beta-1}} \left(2^{\frac{1}{\beta}} r^{\frac{2}{\beta}} + 2^{\frac{\beta-1}{\beta}} (2^{\frac{1-\beta}{2\beta}} + w(r))^2\right) = 1.$$

Expanding this equation we obtain that,

$$2^{-\frac{1}{\beta}} r^{\frac{2}{\beta}} + \frac{\beta}{\beta-1} w(r) + o(w(r)) = 0,$$

which implies that

$$\lim_{r \rightarrow 0} \frac{w(r)}{r^{\frac{2}{\beta}}} = -\frac{\beta-1}{\beta} 2^{-\frac{3\beta+1}{2\beta}}.$$

Therefore, as $r \rightarrow 0$ we have,

$$f' = 2^{\frac{1-\beta}{2\beta}} r^{-\frac{1}{\beta}} - \frac{\beta-1}{\beta} 2^{-\frac{3\beta+1}{2\beta}} r^{\frac{1}{\beta}} + o(r^{\frac{1}{\beta}}).$$

Q.E.D.

The above theorem describes the asymptotic behavior of the rotationally symmetric β -symplectic critical surface as $r \rightarrow 0$ and $r \rightarrow \infty$ for each fixed β . Next we will examine the behavior on a fixed compact set when β goes to infinity and β tends to 0.

Proposition 6.4. $f' \rightarrow 0$ as $\beta \rightarrow \infty$ in $[\varepsilon, +\infty)$ for any $\varepsilon > 0$.

Proof: Using the fundamental inequality $(1+x)^p \geq 1+px$, for $x > 0$ and $p > 0$, we have that $(1+2(f')^2)^{\frac{\beta-1}{2}} \geq 1+(\beta-1)(f')^2$ for $\beta > 1$. By (6.9), we have $f' \leq \sqrt[3]{\frac{1}{(\beta-1)r}}$. This clearly implies the theorem. Q.E.D.

Corollary 6.5. Let Σ_β be a family of smooth complete β -symplectic critical surface which are rotationally symmetric. Suppose $\Sigma_\beta \cap K \neq \emptyset$ for some compact set $K \subset \mathbb{C}^2$ for all β . Then there exists a subsequence Σ_{β_i} , such that Σ_{β_i} converges to a plane locally on $\mathbb{R}^4 \setminus \{0\}$.

Proof: For any $[a, b] \subset (0, \infty)$, we see from Theorem 6.3 that f' converges to 0 uniformly on $[a, b]$ as $\beta \rightarrow \infty$. Therefore, by our assumption, there is a subsequence Σ_{β_i} , which converges to a piece of plane uniformly on $[a, b]$ as β_i tends to infinity. Then the corollary follows from diagonal argument. Q.E.D.

Now we study the limiting surface as $\beta \rightarrow 0$.

Proposition 6.6. *For any $B > A > \sqrt{2}$, the solution f_β of (6.9) with $\beta > 0$ converges uniformly on $[A, B]$ to the catenoid when β goes to zero. And for each fixed $r_0 \in (0, \sqrt{2}]$, we have $\lim_{\beta \rightarrow 0} f'_\beta(r_0) = \infty$.*

Proof: We can rewrite (6.9) as

$$(6.10) \quad \frac{(f'_\beta)^2}{(1 + 2(f'_\beta)^2)^{1-\beta}} = \frac{1}{r^2}.$$

On the interval $[A, B]$, we have $\frac{1}{r^2} \leq \frac{1}{A^2} < \frac{1}{2}$. Using the fundamental inequality $(1 + x)^p \leq 1 + x^p$, for $x > 0$ and $0 < p < 1$, we know from (6.10) and Young's inequality that

$$A^2(f'_\beta)^2 \leq 1 + 2^{1-\beta}(f'_\beta)^{2(1-\beta)} \leq 1 + 2(f'_\beta)^{2(1-\beta)} \leq 2(1 - \beta)(f'_\beta)^2 + 1 + 2\beta.$$

In particular, for $\beta \in [0, 1)$, we have

$$(f'_\beta)^2 \leq \frac{3}{A^2 - 2}.$$

Therefore, we can letting β goes to zero in (6.10) on $[A, B]$ and the first conclusion follows from the continuous dependence of ODE on the parameters.

For the second conclusion, we note that for each fixed $r_0 \in (0, \sqrt{2})$, we have from (6.10) that

$$(6.11) \quad r_0^2(f'_\beta)^2(r_0) \geq (1 + 2(f'_\beta(r_0))^2)^{1-\beta}.$$

Suppose $\liminf_{\beta \rightarrow 0} f'_\beta(r_0) = D < \infty$, then there exists a sequence $\beta_i \rightarrow 0$, such that $\lim_{\beta_i \rightarrow 0} f'_{\beta_i}(r_0) = D$. Replace β by β_i in (6.11) and letting $i \rightarrow \infty$ yields

$$0 \geq 1 + (2 - r_0^2)D^2 \geq 1 \quad \forall \text{ fixed } r_0 \in (0, \sqrt{2}],$$

which gives the desired contradiction. Q.E.D.

In all the previous analysis, we choose $C_1 = C_2 = 1$ in (6.5). Of course, we can also choose $C_1 = C_2 = -1$. Actually, a smooth complete catenoid consists of two pieces of graphs with $C_1 = C_2 = 1$ (denoted by Σ_0^+) and with $C_1 = C_2 = -1$ (denoted by Σ_0^-). By Theorem 6.3, we see that either $C_1 = C_2 = 1$ or $C_1 = C_2 = -1$ produces a complete β -symplectic critical surface Σ_β^+ (or Σ_β^-).

Proposition 6.6 tells us that, the two families of complete β -symplectic critical surfaces Σ_β^+ (or Σ_β^-) converges locally uniformly to the half of catenoid Σ^+ (or Σ^- , respectively) when $\beta \rightarrow 0$. (This can also be seen from the Figure 1 in the following.)

Another interesting observation is the small perturbation Σ_β of the catenoid Σ_0 for $\beta > 0$ consists of two complete β -symplectic critical surfaces Σ_β^+ and Σ_β^- . Note

that the catenoid Σ_0 is not a stable minimal surface, so one may get two β -symplectic critical surfaces in a neighborhood of Σ_0 . It is different from the case that in Theorem 5.1, where we show that there is only one stable β -symplectic critical surface in a neighborhood of a stable minimal surface.

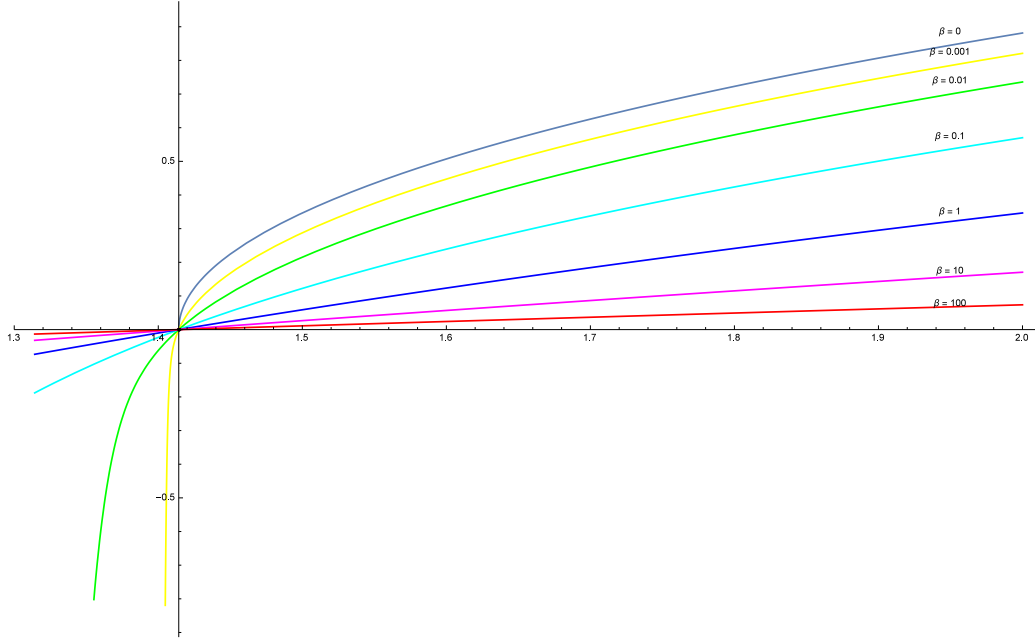


FIGURE 1. Comparison of different values of β

Corollary 6.7. *Let Σ be a complete rotationally symmetric graphic β -symplectic critical surface in \mathbb{C}^2 with $\cos \alpha \geq \delta > 0$ for some positive number δ and $\beta > 0$, then Σ must be a plane.*

Proof: Theorem 6.1 and Theorem 6.3 show that the domain of the defining functions $(f(r), g(r))$ must contain $(0, \infty)$. Furthermore, the asymptotic expansion show that if Σ is not a plane, then either $\lim_{r \rightarrow 0} |f'(r)| = \infty$ or $\lim_{r \rightarrow 0} |g'(r)| = \infty$. By (6.2), we must have $\lim_{r \rightarrow 0} \cos \alpha = 0$. Q.E.D.

Finally, we prove a Liouville theorem for β -symplectic critical surfaces in \mathbb{C}^2 .

Theorem 6.8. *Let Σ^2 be a complete β -symplectic surface in \mathbb{C}^2 with $\cos \alpha \geq \delta > 0$ and $\text{Area}(B^\Sigma(s)) \leq Cs^2$ for $s > 0$, then Σ is a holomorphic curve with respect to some compatible complex structure in \mathbb{C}^2 . Here, $B^\Sigma(s)$ is the intrinsic ball of Σ .*

Proof: From Corollary 2.4, we can easily see that:

$$\begin{aligned} \Delta \frac{1}{\cos \alpha} &= -\frac{\Delta \cos \alpha}{\cos^2 \alpha} + \frac{2|\nabla \cos \alpha|^2}{\cos^3 \alpha} \\ &= \frac{2 \cos^4 \alpha - 2\beta \sin^4 \alpha}{\cos^3 \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)} |\nabla \alpha|^2 + \frac{2 \sin^2 \alpha}{\cos^3 \alpha} |\nabla \alpha|^2 \end{aligned}$$

$$= \frac{2}{\cos \alpha (\cos^2 \alpha + \beta \sin^2 \alpha)} |\nabla \alpha|^2.$$

Since $\cos \alpha \geq \delta > 0$, we have $1 \leq \frac{1}{\cos \alpha} \leq \frac{1}{\delta}$, which means that $\frac{1}{\cos \alpha}$ is a subharmonic function bounded from above on Σ . However, the quadratic area growth implies that Σ is parabolic ([2]). This forces $\frac{1}{\cos \alpha}$ to be a constant on Σ , which implies that Σ is holomorphic with respect to some compatible complex structure in \mathbb{C}^2 . Q.E.D.

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